Invariance Entropy for Control Systems

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Introduction

This introduction is divided into three parts. First we give a short overview of the main aspects and results of the present thesis. Then we explain the concept of topological entropy and how it can be adapted to control systems. In particular, we introduce the central notion of invariance entropy. Finally, we summarize the contents of the individual chapters of this thesis.

Outline of the thesis

The increasing relevance of control systems with restricted digital communication channels has spurred interest in the information rate necessary to accomplish control tasks. The minimal information rate necessary for stabilizing a linear control system has been considered in Nair & Evans [39, 40, 41], Baillieul [7], Hespanha & Ortega & Vasudevan [28], and in Tatikonda & Mitter [51] both for deterministic and stochastic systems under different assumptions on the coding and control scheme. Nevertheless, the same minimal rate was obtained in all of these papers depending only on the unstable eigenvalues of the open-loop system. More results of this kind can be found in the textbook Matveev & Savkin [38]. This suggests that it should be possible to assign an intrinsic quantity to a control system, depending only on the open-loop dynamics, which captures that minimal data rate. For discrete-time systems the first approach to define such a quantity was made by Nair & Evans & Mareels & Moran [42] in 2004. They introduced *topological feedback entropy* as a measure for the inherent rate at which a control system generates stability information and proved that the minimal data rate necessary to stabilize the system into a compact set is exactly given by that measure. In the present thesis, we consider the problem of keeping a continuous-time control system in a compact controlled invariant subset Q of the state space. We introduce the notion of *invariance entropy* which measures how often open-loop control functions have to be updated in order to accomplish this control task. Since the definition of invariance entropy makes no reference to particular feedback strategies, this quantity is intrinsic; in fact, it is invariant under state equivalence. The similarity to the notion of *topological entropy* (for uncontrolled dynamical systems) makes it possible to adapt several techniques used for the computation of the latter and to derive a couple of analogous results. In particular, certain estimates from below and above involving volume growth rates and asymptotic Lipschitz constants, respectively, can be adapted quite easily. For linear systems we show that the invariance entropy is given by the sum of the unstable eigenvalues and hence coincides with the topological entropy of the corresponding linear flow. This result can be generalized for inhomogeneous bilinear systems in form of an estimate from below. Here the unstable eigenvalues are replaced by the unstable minimal Lyapunov exponents of the associated homogeneous (bilinear) system on certain invariant subspaces of the extended state space, i.e., the state space of the corresponding control flow. If the system is control-affine and the property of controllability is imposed on the set Q—more precisely, if Q is the closure of a control set—, the invariance entropy is bounded from above by the sum of the unstable Lyapunov exponents of an arbitrary periodic solution in the interior of Q, provided that the linearization along this solution is controllable. A similar result, proved by Nair et al. [42], holds for the topological feedback entropy, and our proof is a modification of theirs. For one-dimensional locally accessible systems with one control vector field we show that the invariance entropy of a control set equals the minimal Lyapunov exponent of the stationary solutions in the set Q, provided that this minimum is positive, and otherwise vanishes. This result is applied to a controlled linear oscillator with damping. Here we compute the invariance entropy of that subset of the state space, where stabilization at the unstable equilibrium is possible. Then we derive an alternative characterization of the invariance entropy, which reveals the similarity to the notion of topological feedback entropy. Moreover, it enables us to show that the invariance entropy of Q equals the minimal data rate necessary to render Q invariant by a causal coding and control law, in an analogous way as it is proved for the topological feedback entropy in [42]. Finally, we use the alternative characterization to describe an algorithm which computes rigorous upper bounds of the invariance entropy.

Central ideas of the thesis

In the following, we briefly describe the concept of topological entropy for dynamical systems and how it can be adapted to control systems.

Topological entropy measures the rate at which a dynamical system generates information about the initial state, or more loosely speaking, how chaotic the system behaves. In 1965, Adler, Konheim, and McAndrew introduced the concept of topological entropy for a continuous map $f : X \to X$ on a compact topological space X. Prior to that, entropy was already defined in a measure-theoretic setting by Kolmogorov [34] (*metric entropy*), and Adler et al. followed that approach closely. Precisely, their definition is as follows. First, the entropy $H(\mathcal{U})$ of an open cover $\mathcal{U}$ of X is defined as the logarithm of the cardinality of a minimal subcover. Then, for each natural number n an open cover $\mathcal{U}_n$ of X is defined as the collection of all sets $A \subset X$ with the property that every point $x \in A$ has the same trajectory with respect to the cover $\mathcal{U}$ up to time $n-1$, i.e., $x \in A_0, f(x) \in A_1, \ldots, f^{n-1}(x) \in A_{n-1}$ for a fixed sequence $A_0, A_1, \ldots, A_{n-1} \in \mathcal{U}$. The topological entropy $h_{\mathrm{top}}(f,\mathcal{U})$ of f with respect to the cover $\mathcal{U}$ is the limit $\lim_{n\to\infty}\frac{1}{n}H(\mathcal{U}_n)$, and the topological entropy $h_{\text{top}}(f)$ of the map f is defined by taking the supremum over all open covers,

$$h_{\text{top}}(f) := \sup_{\mathcal{U}} h_{\text{top}}(f,\mathcal{U}) = \sup_{\mathcal{U}} \lim_{n\to\infty} \frac{1}{n} H(\mathcal{U}_n).$$

In 1970, Bowen [9] and independently, Dinaburg [19], found two alternative characterizations of the topological entropy for a map on a compact metric space (X,d), based on the notions of "(n,ε)-separated" and "(n,ε)-spanning" sets. Using a family of "dynamic metrics", given by

$$d_{n,f}(x,y) = \max_{0\leq i\leq n-1} d(f^i(x), f^i(y)), \quad n \geq 1,$$

an (n,ε)-separated set $E \subset X$ is a set with the property that $d_{n,f}(x,y) > \varepsilon$ holds for any choice of two different points $x,y \in E$. That is, two points in E can be distinguished if one knows their orbits up to time $n-1$ and if measurements of the state can be made precise enough in order to distinguish between points whose distance is greater than ε. In contrast, an (n,ε)-spanning set $F \subset X$ is a set such that for every $x \in X$ there is at least one $y \in F$ with $d_{n,f}(x,y) \leq \varepsilon$, i.e., the ε-balls with respect to the metric $d_{n,f}$, centered at the points in F, form a cover of X. If one denotes the cardinality of a maximal (n,ε)-separated set by $r_{\text{sep}}(n,\varepsilon,f)$, and by $r_{\text{span}}(n,\varepsilon,f)$ that of a minimal (n,ε)-spanning set, one can consider the exponential growth rates $h_{\text{sep}}(\varepsilon,f)$ and $h_{\text{span}}(\varepsilon,f)$ of these numbers as n tends to infinity. The limits of these growth rates for ε tending to zero exist and coincide with the topological entropy $h_{\text{top}}(f)$:

$$h_{\text{top}}(f) = \lim_{\varepsilon\searrow 0} \limsup_{n\to\infty} \frac{1}{n} \log r_{\text{sep}}(n,\varepsilon,f) = \lim_{\varepsilon\searrow 0} \limsup_{n\to\infty} \frac{1}{n} \log r_{\text{span}}(n,\varepsilon,f).$$

These characterizations made estimation and computation of topological entropy much easier, and they also allowed to extend the concept to uniformly continuous maps and flows on noncompact metric spaces (see Bowen [10]).

Another key result on topological entropy, which was already conjectured in Adler & Konheim & McAndrew [2], is the so-called *variational principle*, which states that the topological entropy equals the supremum over the measure-theoretic entropies with respect to all invariant Borel measures of the given dynamical system. This was proved by Goodwyn [25], Dinaburg [19], and Goodman [24] in the years from 1969 to 1971. Since these days a vast theory of topological entropy has been developed. We refer to Katok & Hasselblatt [33] and other books on dynamical systems for further reading.

In this thesis, we are concerned with the minimal information rate necessary to stabilize a control system in the sense that trajectories do not leave a certain subset of the state space. A first approach to measure this rate using an entropy-like quantity was made by Nair & Evans & Mareels & Moran [42] in 2004, using similar open cover techniques as Adler, Konheim, and McAndrew. They introduced the notion of *(weak and strong) topological feedback entropy* as a measure for the inherent rate at which a discrete-time control system generates "stability information". Precisely, they consider a control system of the form

$$x_{k+1} = F(x_k, u_k)$$

on a topological space X with controls u_k taken from an arbitrary set U. Then the control task of stabilizing the system into a compact set $K \subset X$ is considered, whereas two different invariance conditions of increasing strength are imposed on K. The first one, called *weak invariance*, requires the existence of a compact set K' in the interior of K and a time $n \geq 1$ such that for every $x \in K$ there is a control sequence, which allows to steer from x into the interior of K' in time n. The second one, called *strong invariance*, requires the same in time $n = 1$. Depending on which of these invariance conditions is imposed on K, the weak or strong topological feedback entropy of K is defined. In the strong version, triples (α, G, τ) are considered, where α is an open cover of K, τ a positive integer and G a function, assigning to each $A \in \alpha$ a control sequence $G(A) = (G_k(A))_{k=0}^{\tau-1}$ such that starting from any $x_0 \in A$ with the control sequence $G(A)$ one stays in $\operatorname{int} K'$ up to time τ, i.e., $x_k \in \operatorname{int} K'$ for $k = 1, \ldots, \tau$ with $x_k = F(x_{k-1}, G_{k-1}(A))$. Then for any sequence $A_0, A_1, \ldots, A_j \in \alpha$ an open set $B_j = B_j(A_0, A_1, \ldots, A_j) \subset K$ can be defined containing all $x_0 \in X$ such that the sequence (x_k) given by $x_{k+1} = F(x_k, u_k)$ with $u_k = G(A_{i-1})$ for $k = (i-1)\tau, \ldots, i\tau$, satisfies $x_{i\tau} \in A_i$ for $i = 0, 1, \ldots, j$. The sets B_j form again an open cover β_j of K, which by compactness has a minimal finite subcover, whose cardinality is denoted by $N(\beta_j|K)$. Then the strong topological feedback is given by

$$h^{\mathrm{si}}(F, K, U) := \inf_{(\alpha,G,\tau)} \lim_{j\to\infty} \frac{\log N(\beta_j|K)}{j\tau},$$

where the infimum is taken over all triples (α, G, τ) with the described property. The weak topological feedback entropy is defined analogously. Nair et al. show that the system can be stabilized into K if and only if the data rate in the feedback loop exceeds the topological feedback entropy of K. Moreover, a local version of the topological feedback entropy is defined which measures the minimal data rate necessary to achieve local uniform asymptotic stabilization at a fixed point. Finally, it is proved that for a continuously differentiable system in Euclidean space the local topological feedback entropy can be expressed in terms of the unstable eigenvalues of the fixed point Jacobian.

In contrast to the authors of [42], our approach is closer to the Bowen-Dinaburg characterization of topological entropy via spanning sets. For a continuous-time control system

$$\dot{x}(t) = F(x(t), u(t)), \quad u \in \mathcal{U},$$

on a smooth manifold we consider a pair (K, Q) of compact subsets of the state space with $K \subset Q$ and Q being controlled invariant. The latter means that from every point of Q an admissible trajectory emanates, which remains in Q for all positive times. We call a set $\mathcal{S}$ of admissible control functions T-spanning for (K, Q) if for all $x \in K$ there is some $u \in \mathcal{S}$ such that the solution $\varphi(t, x, u)$ stays in Q for all times $t \in [0, T]$. Then we define the *strict invariance entropy* $h^*_{\mathrm{inv}}(K, Q)$ as the exponential growth rate of the minimal cardinalities

$r^*_{\mathrm{inv}}(T, K, Q)$ of T-spanning sets as T tends to infinity, i.e.,

$$h^*_{\mathrm{inv}}(K, Q) := \limsup_{T\to\infty} \frac{1}{T} \ln r^*_{\mathrm{inv}}(T, K, Q).$$

One inconvenience of this notion is that the numbers $r^*_{\mathrm{inv}}(T, K, Q)$ need not be finite. To avoid this problem, we define another quantity, simply called the *invariance entropy* of (K, Q). To this end, we define (T, ε)-spanning sets for (K, Q) now requiring only that trajectories stay in an ε-neighborhood of Q up to time T. Again the exponential growth rate of the minimal cardinalities $r_{\mathrm{inv}}(T, \varepsilon, K, Q)$ of such sets is considered, now for each positive ε. Then the invariance entropy is introduced as the limit of these growth rates for ε tending to zero,

$$h_{\mathrm{inv}}(K, Q) := \lim_{\varepsilon\searrow 0} \limsup_{T\to\infty} \frac{1}{T} \ln r_{\mathrm{inv}}(T, \varepsilon, K, Q).$$

This definition now avoids the problem of dealing with infinite numbers. Indeed, finiteness of $r_{\mathrm{inv}}(T, \varepsilon, K, Q)$ follows by an easy argument from compactness of K and continuous dependence on initial conditions. Moreover, the definition of $h_{\mathrm{inv}}(K, Q)$ is closer to the characterization of topological entropy via spanning sets, which allows us to adapt several techniques used in the computation of topological entropy. It can also be shown that $h_{\mathrm{inv}}(K, Q)$ is finite in general, which can be disproved for $h^*_{\mathrm{inv}}(K, Q)$ by counterexamples. While $h_{\mathrm{inv}}(K, Q)$ is always a lower bound of $h^*_{\mathrm{inv}}(K, Q)$, it is not clear if equality holds in case that the latter happens to be finite.

Summary of the thesis

In the following, we give a short overview of the contents of each chapter.

In Chapter 1, we develop the basic theory of continuous-time control systems on smooth manifolds. In particular, existence and uniqueness of solutions, continuous and differentiable dependence on initial data, the *Liouville Formula* and the *Wazewski Inequality* are proved. Also linearizations along controlled trajectories and approximations of arbitrary trajectories by trajectories corresponding to simple control functions are discussed. Moreover, some parts of the qualitative theory as developed in Colonius & Kliemann [16] are presented.

In the second chapter, we introduce the central notions of the thesis, namely strict invariance entropy and invariance entropy, and we derive basic properties of these quantities. It is shown that the invariance entropy shares several properties with topological entropy. In particular, it is proved that it is preserved under state equivalence, i.e., under a continuous change of coordinates in the state space. The concept of invariance entropy is then extended to certain noncompact controlled invariant subsets of the state space. Finally, we analyze how the behavior of the control system in a neighborhood of the controlled invariant set Q is related to the invariance entropy of Q. In particular, we consider the opposite situations that Q is isolated and that controllability holds on a neighborhood of Q, respectively.

The third chapter deals with analytical estimates of the invariance entropy. Using similar methods as applied in the estimation of topological entropy, we compute an upper bound for $h_{\rm inv}(K,Q)$ in terms of the maximal eigenvalue of the symmetrized covariant derivative of the right-hand side (with respect to some Riemannian metric) and the fractal dimension of the set K. This bound in particular guarantees finiteness of $h_{\rm inv}(K,Q)$. Using volume growth arguments, a lower bound in terms of the divergence of the right-hand side (with respect to some volume form) is derived in case K has positive volume. Both estimates are applied to projected bilinear control systems on the unit sphere. Finally, the notion of a *uniformly expanding system* is introduced for systems whose solution maps are uniformly expanding for all control functions with respect to a metric on the state space. For a system which is uniformly expanding on Q we compute a lower bound for $h_{\rm inv}(K,Q)$, which is positive if K has positive fractal dimension. For one-dimensional linear systems we obtain a formula for $h_{\rm inv}(K,Q)$, using this estimate together with the aforesaid upper bound.

Chapter 4 is devoted to the study of linear systems and of control sets for control-affine systems. In both cases it turns out that the invariance entropy is closely related to the positive Lyapunov exponents of the system on the set Q. For a linear system the Lyapunov exponents are exactly the real parts of the eigenvalues of the system matrix. Under the assumption that K has positive Lebesgue measure, we show that $h_{\rm inv}(K,Q)$ is given by the sum of those real parts which are positive. For inhomogeneous bilinear systems we are able to compute a lower bound for $h_{\rm inv}(K,Q)$ in terms of the positive minimal Lyapunov exponents on certain invariant subbundles for the control flow of the homogeneous (bilinear) system. For control-affine systems with compact and convex control range it can be shown that the closure of a control set is controlled invariant. Hence, the closure of a relatively compact control set D is a perfect candidate for the controlled invariant set Q in the definition of invariance entropy. Using approximate controllability in D we show that both $h_{\rm inv}(K,\operatorname{cl} D)$ and $h^*_{\rm inv}(K,\operatorname{cl} D)$ are independent of the choice of K, as long as K has nonvoid interior and is contained in D. This observation is the key for a couple of results: For one-dimensional locally accessible control-affine systems with one control vector field we derive an exact formula for the invariance entropy of a control set D. Precisely, we prove that $h_{\rm inv}(K,\operatorname{cl} D)$ and $h^*_{\rm inv}(K,\operatorname{cl} D)$ coincide and equal the maximum of zero and the minimal Lyapunov exponent of the stationary solutions in $\operatorname{cl} D$. We apply this result to a controlled linearized mathematical pendulum with damping in order to compute the invariance entropy of that state space region, where it is possible to stabilize the pendulum at the unstable position. Also for linear systems we can show that $h_{\rm inv}(K,\operatorname{cl} D) = h^*_{\rm inv}(K,\operatorname{cl} D)$. For control-affine systems in arbitrary dimensions we prove that the strict invariance entropy $h^*_{\rm inv}(K,\operatorname{cl} D)$ of a control set D is bounded from above by the sum of the unstable Lyapunov exponents of an arbitrary periodic solution in $\operatorname{int} D$, provided that the linearization along this solution is controllable.

In Chapter 5, we give an alternative characterization of the strict invariance entropy $h^*_{\rm inv}(Q)$ in terms of so-called invariant coverings of Q, which essentially coincides with the definition of strong topological feedback entropy for discrete-time systems except for some minor technical differences. The alternative characterization allows us to compute the strict invariance entropy for one-dimensional linear systems in case Q is a compact interval. Moreover, we can show that the strict invariance entropy $h^*_{\text{inv}}(Q)$ precisely equals the infimum data rate necessary to render the set Q invariant by a causal coding and control law. Finally, we use the characterization via invariant coverings in order to describe an algorithm which computes rigorous upper bounds of $h^*_{\text{inv}}(Q)$ and is mainly based on another algorithm developed by Froyland & Junge & Ochs [21] for the computation of topological entropy.

In the appendix, we provide the basic facts on differentiable manifolds, topological entropy and fractal dimension, which are used in this thesis. In addition, the appendix contains a compilation of technical lemmas.

Finally, some characteristics of the thesis are pointed out:

- Most non-elementary results from the literature used in the proofs are formulated in the first chapter, in the appendix or in footnotes.
- Most sections end with a list of open questions, which are interesting for future work.
- For simplicity, some statements are not formulated in the most general way possible. For example, manifolds and Riemannian metrics are frequently assumed to be of class C^∞, while certainly less smoothness would be sufficient.
- When working with local coordinates of a manifold, the *Einstein Summation Convention* is not used, but in sums the range of the index is omitted, i.e., instead of $\sum_{i=1}^d$ just $\sum_i$ is written, if d is the dimension of the manifold.

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Augsburg, May 2009 *Christoph Kawan*

Chapter 1

Control Theory

In this chapter, we introduce continuous-time control systems on smooth manifolds.[1] We prove elementary results, as e.g., existence and uniqueness of solutions and continuous and differentiable dependence on initial conditions, but also more advanced results, in particular Liouville's Formula and Wazewski's Inequality. Moreover, we discuss the linearization of control systems along controlled trajectories and we formulate a theorem on the approximation of arbitrary trajectories by trajectories corresponding to simple, i.e., piecewise constant control functions, which follows from results of Grasse & Sussmann [26]. Finally, we give a short (and incomplete) survey of the qualitative theory as developed in Colonius & Kliemann [16].

By a control system we understand a family of ordinary differential equations of the form

$$\dot{x}(t) = F(x(t), u(t)),$$

on a smooth manifold M, parametrized by a set $\mathcal{U}$ of admissible control functions $u : \mathbb{R} \to \mathbb{R}^m$. In control theory it is necessary that one can switch from one control function to another at any time. Hence, one must allow control functions with discontinuities. Therefore, we only require that the elements of $\mathcal{U}$ are essentially bounded and thus locally integrable. This implies that for a fixed control function $u \in \mathcal{U}$ also the right-hand side of the differential equation is only locally integrable in t, and we must apply the theory of Carathéodory differential equations. Since we do not know any textbook which rigorously treats Carathéodory differential equations on manifolds, we develop the theory, as far as we need it, by ourselves.[2] Results for Carathéodory equations in Euclidean space can be found, e.g., in the books Hale [27], Kurzweil [35], Sansone & Conti [48], Siegmund [49], Sontag [50] and Walter [53]. In Aulbach & Wanner [5] Carathéodory differential equations on abstract Banach spaces are investigated.

[1] In this thesis, a smooth manifold always means a connected second-countable Hausdorff topological manifold endowed with a C^∞-differentiable structure. See also Section A.1 of the appendix.

[2] In particular, we restrict ourselves to control systems and do not treat general Carathéodory differential equations on manifolds.

1.1 Preliminaries

In the following, we introduce the notion of locally absolutely continuous curves on smooth manifolds and cite results on Carathéodory differential equations, both of which will be needed in order to define control systems and establish a result on existence and uniqueness of solutions.

Locally Absolutely Continuous Curves

The solutions of a Carathéodory differential equation are in general not continuously differentiable but only locally absolutely continuous. Therefore, we first introduce the notion of locally absolutely continuous curves (cf. Kurzweil [35, Defintion 18.1.1, p. 315]).

1.1.1 Definition:
Let $I \subset \mathbb{R}$ be an interval. A mapping $\eta : I \to \mathbb{R}^d$ is called an **absolutely continuous curve** *if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every finite system $\{[\alpha_1, \beta_1], \ldots, [\alpha_n, \beta_n]\}$ of disjoint subintervals of I the implication*

$$\sum_{i=1}^{n} (\beta_i - \alpha_i) < \delta \quad \Rightarrow \quad \sum_{i=1}^{n} \|\eta(\beta_i) - \eta(\alpha_i)\| < \varepsilon$$

holds, where $\|\cdot\|$ is an arbitrary (but fixed) norm on $\mathbb{R}^d$.[3] *The mapping η is called a* **locally absolutely continuous curve** *if the restriction of η to every compact interval $J \subset I$ is absolutely continuous.*

1.1.2 Remarks:

- Obviously, a locally absolutely continuous curve is continuous.
- By Kurzweil [35, Theorem 18.1.3, p. 316] a curve $\eta : I \to \mathbb{R}^d$ is locally absolutely continuous if and only if every coordinate function $\eta_i : I \to \mathbb{R}$, $i = 1, \ldots, d$, is locally absolutely continuous.

For a proof of the following proposition see Kurzweil [35, Theorem 18.1.22, p. 323].

1.1.3 Proposition:
A locally absolutely continuous curve $\eta : I \to \mathbb{R}^d$ is differentiable almost everywhere[4] *in I.*

Before we introduce locally absolutely continuous curves on smooth manifolds, we give a different characterization of such curves in $\mathbb{R}^d$:

[3] Obviously, the property of absolute continuity does not depend on the chosen norm, since all norms on $\mathbb{R}^d$ are equivalent.

[4] By the term "almost everywhere" (abbreviated by "a.e.") we always mean "Lebesgue almost everywhere", i.e., everywhere except for a set of Lebesgue measure zero.

1.1.4 Proposition:
For a mapping $\eta : I \to \mathbb{R}^d$, defined on an interval $I \subset \mathbb{R}$, the following statements are equivalent:

(i) *η is a locally absolutely continuous curve.*

(ii) *For every $\tau \in I$ there exists a compact interval $J_\tau \subset I$ such that $\tau \in \operatorname{int}_I J_\tau$ (the interior of J_τ with respect to I) and $\eta|_{J_\tau}$ is absolutely continuous.*

Proof:
The implication "(i) $\Rightarrow$ (ii)" is trivial. Hence, we only prove the reverse implication: Assume that (ii) is true and let $J \subset I$ be any compact interval. Then for every $\tau \in J$ there exists a compact interval J_τ with $\tau \in \operatorname{int}_I J_\tau$ such that $\eta|_{J_\tau}$ is absolutely continuous. Since $\tau \in \operatorname{int}_I J_\tau \neq \emptyset$, compactness of J implies that there exist $\tau_1, \ldots, \tau_m \in J$ such that $J \subset \bigcup_{i=1}^m J_{\tau_i}$. Let $\tilde{J}_i := J \cap J_{\tau_i}$ for $i = 1, \ldots, m$. Then J equals the union of the compact intervals $\tilde{J}_i$, $i = 1, \ldots, m$, and (by passing over to smaller intervals if necessary) we may also assume that the intervals $\tilde{J}_i$ intersect only in boundary points.
We have to prove that $\eta|_J$ is absolutely continuous. To this end, let $\varepsilon > 0$ be given and choose for each $i = 1, \ldots, m$ a number $\delta_i = \delta_i(\varepsilon/m) > 0$ according to the absolute continuity of η restricted to $\tilde{J}_i$. Let $\delta := \min_{i=1,\ldots,m} \delta_i$, and consider a finite system $\{[\alpha_1, \beta_1], \ldots, [\alpha_n, \beta_n]\}$ of disjoint subintervals of J with

$$\sum_{j=1}^{n} (\beta_j - \alpha_j) < \delta. \tag{1.1}$$

Define

$$\tilde{J}_{ij} := \tilde{J}_i \cap [\alpha_j, \beta_j], \quad i = 1, \ldots, m, \; j = 1, \ldots, n.$$

Then $\{\tilde{J}_{i1}, \ldots, \tilde{J}_{in}\}$ is a finite system of disjoint (compact) subintervals of $\tilde{J}_i$ for every $i = 1, \ldots, m$, whereas some of the intervals $\tilde{J}_{ij}$ may be empty. We may replace those empty intervals by one-point intervals, and thus assume that $\tilde{J}_{ij} = [\alpha_{ij}, \beta_{ij}]$ for some $\alpha_{ij} \leq \beta_{ij}$. Now (1.1) implies

$$\sum_{j=1}^{n} (\beta_{ij} - \alpha_{ij}) \leq \sum_{j=1}^{n} (\beta_j - \alpha_j) < \delta \leq \delta_i$$

and thus we obtain

$$\sum_{j=1}^{n} \|\eta(\beta_{ij}) - \eta(\alpha_{ij})\| < \frac{\varepsilon}{m} \quad \text{for } i = 1, \ldots, m.$$

The interval $[\alpha_j, \beta_j]$ can be written as the union of the intervals $[\alpha_{ij}, \beta_{ij}]$, $i = 1, \ldots, m$, whereas we may assume that the $\tilde{J}_i$ are ordered and hence

$$\alpha_j = \alpha_{i_1 j} \leq \beta_{i_1 j} = \alpha_{i_2 j} \leq \beta_{i_2 j} = \alpha_{i_3 j} \leq \cdots \leq \beta_{i_r j} = \beta_j$$

for some $r = r(j) \in \{1, \ldots, m\}$ and $i_1, i_2, \ldots, i_r$, depending on j, with $i_{k+1} = i_k + 1$ for $k = 1, \ldots, r-1$. Together with the triangle inequality this implies

$$\sum_{j=1}^{n} \|\eta(\beta_j) - \eta(\alpha_j)\| \leq \sum_{j=1}^{n} \sum_{i=i_1(j)}^{i_{r(j)}} \|\eta(\beta_{i_1(j)j}) - \eta(\alpha_{i_{r(j)}(j)j})\|$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{m} \|\eta(\beta_{ij}) - \eta(\alpha_{ij})\|$$
$$= \sum_{i=1}^{m} \underbrace{\sum_{j=1}^{n} \|\eta(\beta_{ij}) - \eta(\alpha_{ij})\|}_{<\varepsilon/m} < \varepsilon.$$

Hence, (i) follows. □

In view of Proposition 1.1.4, the following definition of locally absolutely continuous curves on manifolds seems to be appropriate.

1.1.5 Definition:
Let M be a d-dimensional smooth manifold and $I \subset \mathbb{R}$ an interval. A mapping $\eta : I \to M$ is called a **locally absolutely continuous curve** *if for every $\tau \in I$ there exists a compact subinterval $J_\tau \subset I$ with $\tau \in \operatorname{int}_I J_\tau$ such that $\eta(J_\tau)$ is contained in the domain V of a chart (ϕ, V) and the composed mapping $\phi \circ \eta : J_\tau \to \mathbb{R}^d$ is absolutely continuous.*[5]

1.1.6 Proposition:

(i) *For $M = \mathbb{R}^d$ the Definitions 1.1.1 and 1.1.5 are equivalent.*

(ii) *Let (ϕ_1, V_1) and (ϕ_2, V_2) be two charts of the d-dimensional smooth manifold M and $J \subset I$ a compact interval with $\eta(J) \subset V_1 \cap V_2$ such that $\phi_1 \circ \eta : J \to \mathbb{R}^d$ is absolutely continuous. Then also $\phi_2 \circ \eta : J \to \mathbb{R}^d$ is absolutely continuous. Hence, Definition 1.1.5 does not depend on the chosen charts.*

Proof:

(i) It is clear that a locally absolutely continuous curve in the sense of Definition 1.1.1 is also locally absolutely continuous in the sense of Definition 1.1.5. (We can use the chart $(\mathrm{id}_{\mathbb{R}^d}, \mathbb{R}^d)$.) The reverse follows from Proposition 1.1.4.

(ii) For every finite system $\{[\alpha_1, \beta_1], \ldots, [\alpha_n, \beta_n]\}$ of disjoint subintervals of J we have

$$\sum_{i=1}^{n} \|\phi_2 \circ \eta(\beta_i) - \phi_2 \circ \eta(\alpha_i)\|$$
$$= \sum_{i=1}^{n} \|(\phi_2 \circ \phi_1^{-1}) \circ \phi_1 \circ \eta(\beta_i) - (\phi_2 \circ \phi_1^{-1}) \circ \phi_1 \circ \eta(\alpha_i)\|.$$

The transition function $\phi_2 \circ \phi_1^{-1} : \phi_1(V_1 \cap V_2) \to \phi_2(V_1 \cap V_2)$ is a C^∞-diffeomorphism. We can extend $\phi_2 \circ \phi_1^{-1}$ to a C^∞-function $\psi : \mathbb{R}^d \to \mathbb{R}^d$ by using a cut-off function which equals 1 on $\phi_1(\eta(J))$ and has compact

[5] In Bullo & Lewis [11] one finds another equivalent definition: A curve $\eta : I \to M$ is locally absolutely continuous if $f \circ \eta : I \to \mathbb{R}$ is locally absolutely continuous for every $f \in C^\infty(M)$.

support. The existence of such a function is guaranteed by Lemma A.3.3. Using the mean value theorem this implies

$$\sum_{i=1}^{n} \|\phi_2 \circ \eta(\beta_i) - \phi_2 \circ \eta(\alpha_i)\|$$
$$\leq \underbrace{\max_{x \in \operatorname{supp} \psi} \|D\psi(x)\|}_{=:c} \sum_{i=1}^{n} \|\phi_1 \circ \eta(\beta_i) - \phi_1 \circ \eta(\alpha_i)\|.$$

Now, for given $\varepsilon > 0$ choose $\delta = \delta(\varepsilon/c)$ according to the absolute continuity of $\phi_1 \circ \eta$ on J. Assume that $\sum_{i=1}^{n}(\beta_i - \alpha_i) < \delta$. Then it follows that

$$\sum_{i=1}^{n} \|\phi_2 \circ \eta(\beta_i) - \phi_2 \circ \eta(\alpha_i)\| \leq c\, \frac{\varepsilon}{c} = \varepsilon,$$

which finishes the proof.

□

As for curves in Euclidean space also for curves on smooth manifolds the following proposition is true.

1.1.7 Proposition:
A locally absolutely continuous curve $\eta : I \to M$ is continuous and almost everywhere differentiable.

Proof:
Continuity is obvious. In order to show differentiability, we cover I with countably many compact intervals J_k, $k \in \mathbb{N}$, such that $\eta(J_k)$ is contained in the domain of some chart (ϕ_k, V_k) and $\phi_k \circ \eta$ is absolutely continuous on J_k. Proposition 1.1.3 implies that each of the restrictions $\eta|_{J_k}$, $k \in \mathbb{N}$, is differentiable almost everywhere in J_k. Consequently, also η is differentiable almost everywhere in I, since every countable union of null sets is itself a null set. □

Carathéodory Differential Equations

Next, we introduce Carathéodory differential equations on open subsets of $\mathbb{R}^d$ and cite results on existence and uniqueness of solutions and on differentiable dependence on the initial value.

1.1.8 Definition:
Let $I \subset \mathbb{R}$ be an interval, D an open subset of $\mathbb{R}^d$ and $f : I \times D \to \mathbb{R}^d$ a function with the following properties:

(i) *$f(\cdot, x) : I \to \mathbb{R}^d$ is Lebesgue measurable for each fixed $x \in D$.*

(ii) *$f(t, \cdot) : D \to \mathbb{R}^d$ is continuous for each fixed $t \in I$.*

Then f is called a **Carathéodory function** *and the equation*

$$\dot{x}(t) = f(t, x(t)) \tag{1.2}$$

is called a **differential equation of Carathéodory type** *or a* **Carathéodory differential equation**. *A solution of* (1.2) *is a locally absolutely continuous curve $\eta : J \to D$, defined on some subinterval $J \subset I$, such that*

$$\dot{\eta}(t) = f(t, \eta(t)) \quad \text{for almost all } t \in J.$$

If $\eta(\tau_0) = x_0$ holds for some $(\tau_0, x_0) \in J \times D$, we say that η is a solution of the initial value problem

$$\dot{x}(t) = f(t, x(t)), \quad x(\tau_0) = x_0. \tag{1.3}$$

1.1.9 Remarks:

- Solutions of Carathéodory equations may also be defined as continuous curves which satisfy the integral equation corresponding to (1.2), i.e.,

  $$\eta(t) = x_0 + \int_{\tau_0}^{t} f(s, \eta(s))ds \quad \text{for all } t \in J.$$

  For instance, in Sontag [50] solutions are defined in this way. In Kurzweil [35, Theorem 18.2.6 and Theorem 18.2.7, pp. 326–327], it is proved that both definitions are equivalent.
- In Kurzweil [35], equations are considered whose right-hand side is defined on some subset of $\mathbb{R} \times \mathbb{R}^d$, which is not necessarily a product of the form $I \times D$.

The next proposition yields sufficient conditions for existence and uniqueness of solutions. It can be found in Sontag [50, Theorem 36, pp. 347–351].

1.1.10 Proposition:
Assume that the right-hand side f of the Carathéodory differential equation (1.2) *has the following properties:*

(i) *For each $x_0 \in D$ there exist a real number $\delta > 0$ and a locally integrable function $\alpha : I \to \mathbb{R}_0^+$ such that the ball $B_\delta(x_0)$ is contained in D and*

$$\|f(t, x) - f(t, y)\| \leq \alpha(t)\|x - y\| \quad \text{for all } t \in I \text{ and } x, y \in B_\delta(x_0). \tag{1.4}$$

(ii) *For each fixed $x_0 \in D$ there is a locally integrable function $\beta : I \to \mathbb{R}_0^+$ with*

$$\|f(t, x_0)\| \leq \beta(t) \quad \text{for almost all } t \in I. \tag{1.5}$$

Then, for each pair $(\tau_0, x_0) \in I \times D$ there is some (nonvoid) subinterval $J \subset I$, open relative to I, and there exists a solution $\eta : J \to D$ of the initial value problem (1.3) *with the following property: If $\xi : J' \to D$ is any other solution of* (1.3)*, where $J' \subset I$, then $J' \subset J$ and $\xi = \eta|_{J'}$. The solution η is called the* **maximal solution** *of the initial value problem* (1.3) *in the interval I.*

The following proposition can be found in Kurzweil [35, Remark 18.4.16, p. 338]. It yields not only existence and uniqueness of solutions but also continuous differentiability with respect to the initial value.

1.1.11 Proposition:
Let $I \subset \mathbb{R}$ be an open interval, $D \subset \mathbb{R}^d$ an open set and $f : I \times D \to \mathbb{R}^d$, $(t,x) \mapsto f(t,x)$, a Carathéodory function. Moreover, suppose that the partial derivatives $\frac{\partial f}{\partial x_i} : I \times D \to \mathbb{R}^d$, $i = 1, \ldots, d$, exist and are Carathéodory functions. Let each function $F \in \{f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_d}\}$ satisfy the following condition: For every $(t_0, x_0) \in I \times D$ there exist $\delta_1, \delta_2 > 0$ such that the set

$$Q := \left\{(t,x) \in \mathbb{R}^{1+d} \ : \ |t - t_0| \le \delta_1, \ \|x - x_0\| \le \delta_2\right\}$$

is contained in $I \times D$ and there exists an integrable function $\rho : [t_0 - \delta_1, t_0 + \delta_1] \to \mathbb{R}$ with

$$\|F(t,x)\| \le \rho(t) \quad \text{for all } (t,x) \in Q.$$

Then the solution $\eta_{(t_0,x_0)}$ for each initial value problem

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0,$$

exists on some open interval $J(t_0, x_0) \subset I$ which contains t_0, and is unique. Furthermore, the function $\Phi : G \to \mathbb{R}^d$, defined by

$$\Phi(t, t_0, x) := \eta_{(t_0,x)}(t), \quad G = \left\{(t, t_0, x) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \ : \ t \in J(t_0, x)\right\},$$

is continuous and continuously differentiable with respect to x. The derivative $\frac{\partial \Phi}{\partial x}(t, t_0, x)$ depends locally absolutely continuous on t. Moreover, for every (t_0, x) it holds that

$$\frac{\partial}{\partial t}\frac{\partial}{\partial x}\Phi(t, t_0, x) = \frac{\partial}{\partial x}\frac{\partial}{\partial t}\Phi(t, t_0, x) \quad \text{for almost all } t \in J(t_0, x).$$

Consequently, the function $t \mapsto \frac{\partial}{\partial x}\Phi(t, t_0, x)$ solves the initial value problem

$$\dot{X}(t) = \frac{\partial f}{\partial x}(t, \Phi(t, t_0, x))X(t), \quad X(t_0) = I,$$

where $I \in \mathbb{R}^{d \times d}$ denotes the identity matrix.

For linear Carathéodory differential equations the usual *variation of constants formula* holds (see Aulbach & Wanner [5, Theorem 2.10, p. 58]):

1.1.12 Proposition:
Let $I \subset \mathbb{R}$ be an interval and $A : I \to \mathbb{R}^{d \times d}$, $b : I \to \mathbb{R}^d$ locally integrable mappings. Then the equation

$$\dot{x}(t) = A(t)x(t) + b(t) \tag{1.6}$$

is a Carathéodory differential equation. The solution $\Phi(t; t_0, x_0)$ of the corresponding initial value problem (1.6), $x(t_0) = x_0$, exists and is unique with

$$\Phi(t; t_0; x_0) = \Lambda(t, t_0)x_0 + \int_{t_0}^{t} \Lambda(t, s)b(s)ds$$

for all $(t, t_0, x_0) \in I \times I \times \mathbb{R}^d$, where $t \mapsto \Lambda(t, t_0) \in \mathrm{Gl}(d, \mathbb{R})$ is the unique solution of the initial value problem

$$\dot{X}(t) = A(t)X(t), \quad X(t_0) = I \in \mathbb{R}^{d \times d}.$$

1.2 Basics of Control Systems

In this section, we introduce continuous-time control systems on smooth manifolds and prove basic results, in particular, about existence and uniqueness of solutions and continuous and differentiable dependence on initial data. We also prove the Liouville Formula and the Wazewski Inequality for solutions of control systems. Moreover, we introduce the linearization of control systems along controlled trajectories and finally, we state a result about the approximation of arbitrary solutions by solutions corresponding to piecewise constant control functions.

1.2.1 Definition:
Let $d, m \in \mathbb{N}$, M a d-dimensional smooth manifold, $U \subset \mathbb{R}^m$ a compact set and $F : M \times \mathbb{R}^m \to TM$ a C^1-mapping such that $F_u := F(\cdot, u) : M \to TM$ is a vector field on M for each fixed $u \in \mathbb{R}^m$ (i.e., $F(x,u) \in T_xM$ for all $x \in M$). Let

$$\mathcal{U} := \{u : \mathbb{R} \to \mathbb{R}^m \mid u \text{ measurable with } u(t) \in U \text{ a.e.}\}.$$

Then the family of differential equations

$$\dot{x}(t) = F(x(t), u(t)), \quad u \in \mathcal{U}, \tag{1.7}$$

is called a **control system**, *and $\mathcal{U}$ is called the family of* **admissible control functions**. *The manifold M is called the* **state space** *and the function F the* **right-hand side** *of the control system. For each fixed $u \in \mathcal{U}$ and $x_0 \in M$*

$$\dot{x}(t) = F(x(t), u(t)), \quad x(0) = x_0, \tag{1.8}$$

is called the **initial value problem** *for the pair (u, x_0). A* **solution** *of (1.8) is a locally absolutely continuous curve $\eta : I \to M$, defined on an interval I with $0 \in I$, such that $\eta(0) = x_0$ and*

$$\dot{\eta}(t) = F(\eta(t), u(t)) \quad \text{for almost all } t \in I.$$

1.2.2 Remark:
Actually we do not need the differentiability of the right-hand side F in (1.7) with respect to the second argument in order to establish results on existence and uniqueness of solutions.[6] In Grasse & Sussmann [26] one finds a more general definition for control systems on manifolds, where it is only required that F is continuous and every local representation of F is continuously differentiable in the first argument. Moreover, the control range U is an arbitrary separable metric space. Later in this section, this definition will be explained in greater detail.

1.2.3 Notation:
For a control system on an open subset M of $\mathbb{R}^d$ we write $D_xF(x,u)$ or $D_1F(x,u)$ for the partial derivative of F by the first argument, and $D_uF(x,u)$ or $D_2F(x,u)$ for the partial derivative by the second argument. Note that $D_xF(x,u) \in \mathbb{R}^{d \times d}$ and $D_uF(x,u) \in \mathbb{R}^{d \times m}$.

[6] But we will need the differentiability in the second argument in order to linearize the system along controlled trajectories.

Existence and Uniqueness of Solutions

The following theorem guarantees existence and uniqueness of solutions for the initial value problem (1.8).

1.2.4 Theorem and Definition:
Consider the control system (1.7) *and let* $(u, x_0) \in \mathcal{U} \times M$. *Then there exists a solution* $\eta : I \to M$ *of the initial value problem* (1.8), *defined on a (nonvoid) open interval* I, *with the following property: If* $\xi : J \to M$ *is another solution of* (1.8), *then* $J \subset I$ *and* $\xi = \eta|_J$. *The solution* η *is denoted by*

$$\varphi(\cdot, x_0, u) : I \to M \tag{1.9}$$

and it is called the **maximal solution** *of the initial value problem* (1.8). *The interval* I *is also denoted by* $I_{\max}(u, x_0)$.

Proof:
The proof is subdivided into three steps.

Step 1: We show that there exists a solution of (1.8) defined on a small open interval J with $0 \in J$. To this end, choose a chart (ϕ, V) of M with $x_0 \in V$. Let $W := \phi(V) \subset \mathbb{R}^d$ and define $\tilde{F} : W \times \mathbb{R}^m \to \mathbb{R}^d$ by

$$\tilde{F}(y, u) := D\phi_{\phi^{-1}(y)} F(\phi^{-1}(y), u) \quad \text{for all } (y, u) \in W \times \mathbb{R}^m. \tag{1.10}$$

Note that $T_y\mathbb{R}^d$ can be identified canonically with $\mathbb{R}^d$ for all $y \in W$ and thus we may assume that $\tilde{F}$ maps to $\mathbb{R}^d$ instead of $T\mathbb{R}^d$. By construction, $\tilde{F}$ is a C^1-function. It can be decomposed in the following way:

$$W \times \mathbb{R}^m \xrightarrow{\phi^{-1} \times \mathrm{id}_{\mathbb{R}^m}} V \times \mathbb{R}^m \xrightarrow{F} TV \xrightarrow{D\phi} TW \cong W \times \mathbb{R}^d \xrightarrow{\pi_2} \mathbb{R}^d.$$

Here $\pi_2 : W \times \mathbb{R}^d \to \mathbb{R}^d$ is the projection onto the second factor. For the fixed control function $u \in \mathcal{U}$ define

$$f : \mathbb{R} \times W \to \mathbb{R}^d, \quad f(t, y) := \tilde{F}(y, u(t)) \quad \text{for all } (t, y) \in \mathbb{R} \times W$$

and consider the differential equation

$$\dot{y}(t) = f(t, y(t)). \tag{1.11}$$

We want to show that f is a Carathéodory function. To this end, consider for fixed $y \in W$ the function $f(\cdot, y) : \mathbb{R} \to \mathbb{R}^d$, $t \mapsto \tilde{F}(y, u(t))$. This function is measurable, since it can be written as the composition of measurable functions:

$$t \mapsto u(t) \mapsto (y, u(t)) \mapsto \tilde{F}(y, u(t)).$$

Now fix $t \in \mathbb{R}$. Then $x \mapsto f(t, x) = \tilde{F}(x, u(t))$ is continuous, since $\tilde{F}$ is continuous. This proves that (1.11) is a differential equation of Carathéodory type. We want to show that there exists a unique solution of the initial value problem

$$\dot{y}(t) = f(t, y(t)), \quad y(0) = \phi(x_0).$$

To this end, we show that f satisfies the conditions (1.4) and (1.5) in Proposition 1.1.10: Fix $y_0 \in W$ and choose $\delta > 0$ small enough that $\operatorname{cl} B_\delta(y_0) \subset W$. Then the mean value theorem guarantees that

$$\begin{aligned} \|f(t,y_1) - f(t,y_2)\| &= \|\tilde{F}(y_1,u(t)) - \tilde{F}(y_2,u(t))\| \\ &\leq \max_{(y,v)\in \operatorname{cl} B_\delta(y_0)\times U} \left\| D_y \tilde{F}(y,v) \right\| \cdot \|y_1 - y_2\| \end{aligned}$$

for almost all $t \in \mathbb{R}$ and all $y_1, y_2 \in B_\delta(y_0)$, since $B_\delta(y_0)$ is convex. This proves (1.4). Condition (1.5) holds, since

$$\|f(t,y_0)\| = \|\tilde{F}(y_0,u(t)\| \leq \max_{v\in U} \|\tilde{F}(y_0,v)\|$$

for almost all $t \in \mathbb{R}$. (Recall that $u(t) \in U$ for almost all $t \in \mathbb{R}$.) Consequently, Proposition 1.1.10 yields a maximal solution $\tilde{\eta} : J \to W$ on a nonvoid open interval J satisfying $\tilde{\eta}(0) = \phi(x_0)$. We define $\eta : J \to M$ by

$$\eta(t) := \phi^{-1} \circ \tilde{\eta}(t) \quad \text{for all } t \in J.$$

Then η clearly is a locally absolutely continuous curve on M with $\eta(0) = x_0$, and it holds that

$$\begin{aligned} \dot{\eta}(t) &= D\phi^{-1}_{\tilde{\eta}(t)} \dot{\tilde{\eta}}(t) = D\phi^{-1}_{\tilde{\eta}(t)} \tilde{F}(\tilde{\eta}(t),u(t)) \\ &= D\phi^{-1}_{\phi(\eta(t))} D\phi_{\eta(t)} F(\eta(t),u(t)) = F(\eta(t),u(t)) \end{aligned}$$

for almost all $t \in J$. Hence, η is a solution of the initial value problem (1.8).

Step 2: We show that any two solutions $\xi_i : J_i \to M$, $i = 1, 2$, of (1.8), defined on open intervals J_1 and J_2, respectively, coincide on $J_1 \cap J_2$. To this end, we consider the set

$$A := \{t \in J_1 \cap J_2 \mid \xi_1(t) = \xi_2(t)\}.$$

Since $0 \in J_1 \cap J_2$, A is nonvoid. By continuity of ξ_1 and ξ_2, A is closed in $J_1 \cap J_2$. To see that A is also open, fix $\tau \in A$ and consider the initial value problem

$$\dot{x}(t) = F(x(t),u(t)), \quad x(\tau) = \xi_1(\tau) = \xi_2(\tau).$$

Then, by the same construction as in Step 1, one obtains a local solution defined on an open interval containing τ. It can easily be shown that this solution is independent of the chosen chart and by Proposition 1.1.10 it is locally unique. This implies that ξ_1 and ξ_2 must coincide in a neighborhood of τ and hence A is open. Since $J_1 \cap J_2$ is connected, we conclude that $A = J_1 \cap J_2$.

Step 3: We prove the assertion: We define I as the union of all open intervals containing 0, on which there exists a solution of (1.8). Then I is nonvoid and open. By definition, for arbitrary $\tau \in I$ we find some interval $I_\tau \subset I$ with $\tau \in I_\tau$, on which a solution ξ is defined. We set $\eta(\tau) := \xi(\tau)$. By Step 2, the so defined function is independent of the individual solutions ξ we use. We clearly have $\eta(0) = x_0$. Moreover, η is locally absolutely continuous, since by definition, locally it coincides with a curve having this property. Obviously, it is also true that η satisfies the differential equation $\dot{x}(t) = F(x(t), u(t))$ almost everywhere. By construction, any other solution $\xi : J \to M$ of (1.8) satisfies $J \subset I$ and $\xi = \eta|_J$. $\square$

The following proposition yields a sufficient condition for a maximal solution to be defined on $\mathbb{R}$.

1.2.5 Proposition:
Let $\eta : I \to M$ be a maximal solution of the control system (1.7)*. Assume that $K \subset M$ is a compact set such that $\eta(t) \in K$ for all $t \in I$. Then $I = \mathbb{R}$.*

Proof:
Let $I = (a, b)$ with $-\infty \leq a < 0 < b \leq \infty$. We prove that $b = \infty$. (The proof for $a = -\infty$ works analogously.) Assume to the contrary that $b < \infty$ and let $(t_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence with $t_n \in (a, b)$ for all $n \in \mathbb{N}$ and $t_n \to b$ for $n \to \infty$. All points of the sequence $(\eta(t_n))_{n\in\mathbb{N}}$ are by assumption elements of the compact set K. Consequently, there exists a converging subsequence. So we may assume that $\eta(t_n) \to z$ for some $z \in K$. Let (ϕ, V) be a chart around z with

$$\phi(z) = 0 \quad \text{and} \quad \operatorname{cl} B_1(0) \subset \phi(V),$$

where $B_1(0)$ denotes the Euclidean ball with radius 1 centered at $0 \in \mathbb{R}^d$. Let $\tilde{F} : \phi(V) \times \mathbb{R}^m \to \mathbb{R}^d$ be defined as in (1.10), i.e., the local version of the control system (1.7) with respect to the chart (ϕ, V) is given by

$$\dot{y}(t) = \tilde{F}(y(t), u(t)), \quad u \in \mathcal{U}.$$

We define

$$L := \max_{(y,u)\in\operatorname{cl}(B_1(0))\times U} \|\tilde{F}(y, u)\|.$$

Since t_n converges to b and $\eta(t_n)$ converges to z, we find $n_0 \in \mathbb{N}$ such that

$$\eta(t_n) \in V,\ \phi(\eta(t_n)) \in B_{1/4}(0) \text{ for all } n \geq n_0 \text{ and } (b - t_{n_0})L < \tfrac{1}{4}.$$

Let $t \in (t_{n_0}, b)$ be any time such that $\eta([t_{n_0}, t]) \subset V$. Since V is open and η is continuous, such t exists. We obtain

$$\begin{aligned}
\|\phi(\eta(t_{n_0})) - \phi(\eta(t))\| &= \left\| \int_{t_{n_0}}^{t} \tilde{F}(\phi(\eta(\tau)), u(\tau)) d\tau \right\| \\
&\leq \int_{t_{n_0}}^{t} \left\| \tilde{F}(\phi(\eta(\tau)), u(\tau)) \right\| d\tau \\
&\leq (t - t_{n_0})L < (b - t_{n_0})L < \tfrac{1}{4}.
\end{aligned}$$

Note that the integral in the preceding estimate exists by Lemma A.3.5. Using the triangle inequality we get

$$\|\phi(\eta(t))\| \leq \|\phi(\eta(t)) - \phi(\eta(t_{n_0}))\| + \|\phi(\eta(t_{n_0}))\| < \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}.$$

Consequently, $\phi(\eta([t_{n_0}, b))) \subset B_{1/2}(0)$ and $\eta([t_{n_0}, b)) \subset V$. For every $t \in (t_{n_0}, b)$ there is some $n = n(t) \in \mathbb{N}$ with $t_n > t$. By the triangle inequality we have

$$\|\phi(\eta(t))\| \leq \|\phi(\eta(t)) - \phi(\eta(t_n))\| + \|\phi(\eta(t_n))\|.$$

For $t \to b$ one has $\|\phi(\eta(t_n))\| \to 0$ and

$$\|\phi(\eta(t)) - \phi(\eta(t_n))\| \leq \int_t^{t_n} \left\|\tilde{F}(\phi(\eta(\tau)), u(\tau))\right\| d\tau \leq (t_n - t)L \to 0.$$

It follows that η can be extended continuously to a solution on $(a, b]$ by $\eta(b) := z$, and consequently also to a solution on some open interval $(a, b+\varepsilon)$ with $\varepsilon > 0$, which contradicts the maximality of $\eta : (a, b) \to M$. □

1.2.6 Corollary:
Assume that the right-hand side F of control system (1.7) *satisfies*

$$F(x, u) = 0 \quad \text{for all } x \in M \backslash K \text{ and } u \in \mathbb{R}^m,$$

where $K \subset M$ is compact. Then all maximal solutions are defined on $\mathbb{R}$.

Proof:
Let $\eta : I \to M$ be a maximal solution. Assume that $\xi(\tau) \in M \backslash K$ for some $\tau \in I$. Since $M \backslash K$ is open in M and ξ is continuous, then $\xi(t) \in M \backslash K$ for all t in some interval of the form $(\tau - \varepsilon, \tau + \varepsilon)$ with $\varepsilon > 0$. It follows that

$$\frac{d}{dt}\eta(t) = F(\eta(t), u(t)) = 0 \text{ for almost all } t \in (\tau - \varepsilon, \tau + \varepsilon).$$

Consequently, η is constant on $(\tau - \varepsilon, \tau + \varepsilon)$ and thus on I. By Proposition 1.2.5 $I = \mathbb{R}$ follows. If there is no $\tau \in I$ with $\xi(\tau) \notin K$, then $\xi(t) \in K$ for all $t \in I$ and hence also $I = \mathbb{R}$ by Proposition 1.2.5. □

Next, we introduce the so-called shift flow on the set $\mathcal{U}$ of admissible control functions. Later, in the context of control-affine systems, we endow $\mathcal{U}$ with a topology which makes $\mathcal{U}$ a compact metrizable space and the shift flow a continuous dynamical system.

1.2.7 Definition:
The **shift flow** $\Theta : \mathbb{R} \times \mathcal{U} \to \mathcal{U}$ *is defined by*

$$\Theta(t, u) := \Theta_t u \quad \text{with } (\Theta_t u)(s) = u(t + s) \text{ for all } s \in \mathbb{R}.$$

It is easy to see that the shift flow is well-defined (i.e., $\Theta_t u \in \mathcal{U}$ for all $(t, u) \in \mathbb{R} \times \mathcal{U}$) and that it satisfies the flow properties: $\Theta(0, u) = u$ and $\Theta(t + s, u) = \Theta(t, \Theta(s, u))$ for all $t, s \in \mathbb{R}$ and $u \in \mathcal{U}$. Now consider the control system (1.7). The maximal solutions $\varphi(\cdot, x, u)$ define a mapping

$$\varphi : \mathcal{D} \to M, \quad (t, x, u) \mapsto \varphi(t, x, u), \tag{1.12}$$

where

$$\mathcal{D} = \{(t, x, u) \in \mathbb{R} \times M \times \mathcal{U} \mid t \in I_{\max}(u, x)\}.$$

The mapping φ satisfies a **cocycle property**, which is described in the following proposition.[7]

1.2.8 Proposition:
If $(u,x) \in \mathcal{U} \times M$ and $s \in I_{\max}(u,x)$, then the following assertions hold:

(i) $I_{\max}(\Theta_s u, \varphi(s,x,u)) = I_{\max}(u,x) - s$.

(ii) *For all $t \in I_{\max}(u,x) - s$ we have*

$$\varphi(t+s,x,u) = \varphi(t,\varphi(s,x,u),\Theta_s u). \tag{1.13}$$

Proof:
Let $I := I_{\max}(u,x) - s$. Then the function

$$\xi : I \to M, \quad \xi(t) := \varphi(t+s,x,u),$$

is well-defined. On $J := I_{\max}(\Theta_s u, \varphi(s,x,u))$ we consider the function

$$\eta : J \to M, \quad \eta(t) := \varphi(t,\varphi(s,x,u),\Theta_s u).$$

Both I and J contain $t_0 = 0$ and

$$\xi(0) = \varphi(s,x,u) = \varphi(0,\varphi(s,x,u),\Theta_s u) = \eta(0).$$

Moreover, ξ and η are locally absolutely continuous and thus differentiable almost everywhere. For the derivative of ξ we obtain

$$\dot{\xi}(t) = \frac{d}{dt}\varphi(t+s,x,u) = F(\varphi(t+s,x,u),u(t+s)) = F(\xi(t),(\Theta_s u)(t))$$

for almost all $t \in I$, and for η we get

$$\begin{aligned}\dot{\eta}(t) &= \frac{d}{dt}\varphi(t,\varphi(s,x,u),\Theta_s u)\\ &= F(\varphi(t,\varphi(s,x,u),\Theta_s u),(\Theta_s u)(t)) = F(\eta(t),(\Theta_s u)(t))\end{aligned}$$

for almost all $t \in J$. Hence, ξ and η are both solutions of the same initial value problem and thus coincide on $I \cap J$, and the maximal interval of definition for both ξ and η is $J = I_{\max}(\Theta_s u, \varphi(s,x,u))$, which implies $I \subset J$. But on the other hand, it is clear that ξ cannot be extended over I, such that $I = J$ must hold. This implies the assertions. $\square$

In many cases we will fix one or two of the arguments of φ. In order to stress which arguments are fixed, we use the following notation:

$$\varphi_{t,u}(x) \equiv \varphi_{x,u}(t) \equiv \varphi_u(t,x) \equiv \varphi_t(x,u) \equiv \varphi(t,x,u).$$

1.2.9 Remark:
Since later we will only consider solutions which do not leave a compact subset of the state space, due to Proposition 1.2.5 we may assume in the following that all maximal solutions are defined on $\mathbb{R}$.

[7] For a general definition of *cocycles* see Rasmussen [46, Definition 2.1, p. 9].

Regularity Properties of the Solution

In this subsection, we prove continuous dependence of the cocycle φ on (t, x) and continuously differentiable dependence on x.

1.2.10 Theorem:
Consider control system (1.7) *and assume that all maximal solutions are defined on* $\mathbb{R}$. *Then, for every control function* $u \in \mathcal{U}$ *the mapping* $\varphi_u : \mathbb{R} \times M \to M$ *is continuous.*

Proof:
First we fix a metric d on M, which induces the given topology.[8] We show continuity of φ_u at an arbitrarily chosen point $(t_*, x_*) \in \mathbb{R} \times M$ in five steps.

Step 1: By the triangle inequality for every $(t, x) \in \mathbb{R} \times M$ we have

$$d(\varphi(t,x,u), \varphi(t_*,x_*,u)) \leq d(\varphi(t,x,u), \varphi(t,x_*,u)) + d(\varphi(t,x_*,u), \varphi(t_*,x_*,u)).$$

Continuity of the single solution $\varphi(\cdot, x_*, u)$ guarantees that the second summand tends to zero as $t \to t_*$. Thus, we only have to show that also the first summand tends to zero as $(t, x) \to (t_*, x_*)$. Without loss of generality we assume that $t_* > 0$.

Step 2: We show that it suffices to consider the case when the set $\varphi([0, t_*], x_*, u)$ is contained in the domain of a chart. To this end, assume that the assertion is true in this case. In general, it is always possible to find finitely many charts

$$(\phi_1, V_1), (\phi_2, V_2), \dots, (\phi_n, V_n)$$

and times

$$0 = t_0 < t_1 < \cdots < t_n = t_*$$

such that the following inclusions hold:

$$\varphi([t_{i-1}, t_i], x_*, u) \subset V_i \quad \text{for } i = 1, 2, \dots, n.$$

Since x_* and $\varphi(t_1, x_*, u)$ are contained in V_1, by our assumption we have

$$\varphi(t,x,u) \to \varphi(t_1,x_*,u) \quad \text{for } (t,x) \to (t_1, x_*). \tag{1.14}$$

Now assume that $(t, x) \to (t_2, x_*)$. Then by (1.14) it follows that

$$(t - t_1, \varphi(t_1, x, u)) \to (t_2 - t_1, \varphi(t_1, x_*, u)).$$

Since $\varphi(t_1, x_*, u)$ and $\varphi(t_2, x_*, u) = \varphi(t_2 - t_1, \varphi(t_1, x_*, u), \Theta_{t_1} u)$ are contained in V_2, by our assumption this implies

$$\varphi(t - t_1, \varphi(t_1, x, u), \Theta_{t_1} u) \to \varphi(t_2 - t_1, \varphi(t_1, x_*, u), \Theta_{t_1} u),$$

[8] Note that every smooth manifold is metrizable, e.g., by the distance induced by a Riemannian metric (see Gallot & Hulin & Lafontaine [22, Theorem 2.2, p. 49]).

which by the cocycle property (1.13) can be written as

$$\varphi(t,x,u) \to \varphi(t_2,x_*,u).$$

By repeating this process one can show that $(t,x,u) \to (t_*,x_*,u)$ implies $\varphi(t,x,u) \to \varphi(t_*,x_*,u)$. Hence, from now on we may assume that $\varphi([0,t_*+c],x_*,u)$ (for small $c>0$) is contained in the domain V of a chart (ϕ,V).

Step 3: We consider the local version of the control system on M with respect to the chart (ϕ,V). The corresponding right-hand side is denoted by $\tilde{F}:\phi(V)\times\mathbb{R}^m \to \mathbb{R}^d$ (cf. (1.10)). We may assume that

$$\phi(V) = B_1(0) \quad \text{and} \quad \phi(x_*) = 0.$$

By continuity of $\varphi(\cdot,x_*,u)$ the set $(\phi\circ\varphi)([0,t_*+c],x_*,u)$ is compact and thus we can find a number $r\in(0,1)$ with

$$\phi\circ\varphi([0,t_*+c],x_*,u) \subset B_r(0) \subset B_1(0).$$

Let $\xi(t) := \phi(\varphi(t,x_*,u))$ for $t\in[0,t_*+c]$. Then, since $\xi(0)=\phi(x_*)=0$, we obtain

$$\xi(t) = \int_0^t \tilde{F}(\xi(\tau),u(\tau))d\tau \quad \text{for all } t\in[0,t_*+c].$$

Now we extend $\tilde{F}$ to a C^1-function on $\mathbb{R}^d\times\mathbb{R}^m$ by choosing a cut-off function $\theta:\mathbb{R}^d\to[0,1]$ with compact support $\operatorname{supp}\theta\subset B_1(0)$ and

$$\theta(y) = 1 \quad \text{for all } y\in B_r(0).$$

The existence of such θ is guaranteed by Lemma A.3.3. We consider the differential equation

$$\dot{y}(t) = \tilde{G}(y(t),u(t)) \tag{1.15}$$

with $\tilde{G}:\mathbb{R}^d\times\mathbb{R}^m\to\mathbb{R}^d$, given by

$$\tilde{G}(y,u) := \begin{cases} \theta(y)\tilde{F}(y,u) & \text{for } y\in B_1(0), \\ 0 & \text{otherwise.} \end{cases}$$

Step 4: We estimate the distance between the solution ξ and solutions of (1.15): For $\tilde{G}$ one gets a global Lipschitz constant with respect to the first variable, namely

$$L = \max_{(y,u)\in\operatorname{supp}\theta\times U} \left\| D_y\tilde{G}(y,u) \right\|.$$

The maximal solution of the initial value problem $y(0)=y$ for equation (1.15) will be denoted by $\eta_y : I_{\max}(u,y)\to\mathbb{R}^d$. By Corollary 1.2.6 we have $I_{\max}(u,y)=\mathbb{R}$ for all $y\in\mathbb{R}^d$. Now we compare these solutions with ξ on the interval $[0,t_*+c]$:

$$\xi(t)-\eta_y(t) = (\underbrace{\phi(x_*)}_{=0}-y) + \int_0^t \left[\tilde{F}(\xi(\tau),u(\tau)) - \tilde{G}(\eta_y(\tau),u(\tau))\right] d\tau. \tag{1.16}$$

Since $\xi(\tau) \in B_r(0)$ for all $\tau \in [0, t_* + c]$, we have $\tilde{F}(\xi(\tau), u(\tau)) = \tilde{G}(\xi(\tau), u(\tau))$ for all $\tau \in [0, t_* + c]$. Consequently, we can replace $\tilde{F}$ by $\tilde{G}$ in equation (1.16), and we obtain for all $t \in [0, t_* + c]$:

$$\begin{aligned} \|\xi(t) - \eta_y(t)\| &\leq \|y\| + \int_0^t \left\| \tilde{G}(\xi(\tau), u(\tau)) - \tilde{G}(\eta_y(\tau), u(\tau)) \right\| d\tau \\ &\leq \|y\| + L \int_0^t \|\xi(\tau) - \eta_y(\tau)\| d\tau. \end{aligned}$$

By Lemma A.3.4 (the Gronwall Lemma) this implies

$$\|\xi(t) - \eta_y(t)\| \leq \|y\| e^{Lt} \leq \|y\| e^{L(t_* + c)} \quad \text{for all } t \in [0, t_* + c]. \tag{1.17}$$

<u>*Step 5:*</u> We show continuity of $\varphi(\cdot, \cdot, u)$ at (t_*, x_*): For given $\varepsilon > 0$ we choose $\delta > 0$ such that $\max\{|t - t_*|, \|y\|\} < \delta$ implies $t \in [0, t_* + c]$ and $\|y\| e^{L(t_* + c)} < \varepsilon$. If we choose $\varepsilon > 0$ small enough, we obtain that $\eta_y(t) \in B_r(0)$ for all $t \in [0, t_* + c]$ and thus

$$\eta_y(t) = y + \int_0^t \tilde{F}(\eta_y(\tau), u(\tau)) d\tau.$$

Then $\phi^{-1} \circ \eta_y : [0, t_* + c] \to M$ is a solution of the original control system on M, namely

$$\phi^{-1} \circ \eta_y(t) = \varphi(t, \phi^{-1}(y), u).$$

Consequently, for $(t, x) \to (t_*, x_*)$ we get $\phi(x) \to \phi(x_*) = 0$ and thus

$$\varphi(t, x, u) = \phi^{-1} \circ \eta_{\phi(x)}(t) \to \phi^{-1} \circ \xi(t_*) = \varphi(t_*, x_*, u).$$

This proves the claim. $\square$

1.2.11 Corollary:
Under the assumptions of Theorem 1.2.10 let $u \in \mathcal{U}$ and $(t_0, x_0) \in \mathbb{R} \times M$ be fixed. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d(x, x_0) < \delta \quad \Rightarrow \quad d(\varphi(t, x, u), \varphi(t, x_0, u)) < \varepsilon \quad \textit{for all } t \in [0, t_0].$$

Proof:
By Lemma A.3.2 there is $\rho > 0$ such that $\operatorname{cl} B_\rho(x_0)$ is compact. Then also $K := [0, t_*] \times \operatorname{cl} B_\rho(x_0)$ is compact. Theorem 1.2.10 implies that $\varphi(\cdot, \cdot, u)$ is uniformly continuous on K. Hence, for every $\varepsilon > 0$ there is $\delta \in (0, \rho)$ such that $d(x, x_0) < \delta$ implies $d(\varphi(t, x, u), \varphi(t, x_*, u)) < \varepsilon$ for all $t \in [0, t_0]$. $\square$

1.2.12 Corollary:
For all $t \in \mathbb{R}$ and $u \in \mathcal{U}$ the map $\varphi_{t,u} : M \to M$ is a homeomorphism with inverse $\varphi_{-t, \Theta_t u}$.

Proof:
Continuity of both $\varphi_{t,u}$ and $\varphi_{-t, \Theta_t u}$ follows from Theorem 1.2.10. From Proposition 1.2.8 and the flow properties of Θ for all $x \in M$ it follows that

$$\begin{aligned} \varphi_{t,u}(\varphi_{-t, \Theta_t u}(x)) &= \varphi(t, \varphi(-t, x, \Theta_t u), u) = \varphi(t, \varphi(-t, x, \Theta_t u), \Theta_{-t} \Theta_t u) \\ &= \varphi(0, x, \Theta_t u) = x \end{aligned}$$

and

$$\varphi_{-t,\Theta_t u}(\varphi_{t,u}(x)) = \varphi(-t,\varphi(t,x,u),\Theta_t u) = \varphi(0,x,u) = x.$$

This proves the claim. □

1.2.13 Lemma:
Consider control system (1.7) and assume that M is an open subset of $\mathbb{R}^d$. Then for each $u \in \mathcal{U}$ the function $(t,x) \mapsto \varphi_{t,u}(x)$ (where defined) is continuously differentiable with respect to x, and for fixed $x \in M$ the function $t \mapsto D\varphi_{t,u}(x)$ satisfies the Carathéodory differential equation

$$\dot{y}(t) = D_1F(\varphi(t,x,u),u(t))y(t)$$

almost everywhere on $I_{\max}(u,x)$. Moreover, the derivative $D\varphi_{t,u}(x)$ depends locally absolutely continuous on t.

Proof:
Fix some $u \in \mathcal{U}$ and let $f : \mathbb{R} \times M \to \mathbb{R}^d$ be defined by $f(t,x) := F(x,u(t))$. Then the assertion follows from Proposition 1.1.11 if we can verify the hypotheses on f: By assumption F is a C^1-function. Hence, the partial derivatives $\frac{\partial f}{\partial x_i}$, $i = 1,\ldots,d$, exist. Moreover, they are Carathéodory functions, since for fixed $x \in D$ we have that $f(\cdot,x) = F(x,u(\cdot))$ and $\frac{\partial f}{\partial x_i}(\cdot,x) = \frac{\partial F}{\partial x_i}(x,u(\cdot))$ are measurable as the composition of measurable functions (see also the proof of Theorem 1.2.4), and for fixed $t \in \mathbb{R}$ the functions $f(t,\cdot) = F(\cdot,u(t))$ and $\frac{\partial f}{\partial x_i}(\cdot,u(t)) = \frac{\partial F}{\partial x_i}(\cdot,u(t))$ are continuous, since F is a C^1-function. Now fix some $(t_0,x_0) \in \mathbb{R} \times M$. Let $\delta > 0$ be chosen such that the compact ball $B \subset \mathbb{R}^d$ with radius δ centered at x_0 is still contained in M. Then for all $x \in B$ and almost all $t \in \mathbb{R}$ it holds that

$$\|f(t,x)\| = \|F(x,u(t))\| \leq \max_{(x,v)\in B\times U} \|F(x,v)\|$$

and

$$\left\|\frac{\partial f}{\partial x_i}(t,x)\right\| = \left\|\frac{\partial F}{\partial x_i}(x,u(t))\right\| \leq \max_{(x,v)\in B\times U} \left\|\frac{\partial F}{\partial x_i}(x,v)\right\|, \quad i = 1,\ldots,d.$$

This proves that the hypotheses on f hold. □

1.2.14 Theorem:
Consider control system (1.7) and assume that all maximal solutions are defined on $\mathbb{R}$. Then for all $t \in \mathbb{R}$ and $u \in \mathcal{U}$ the map $\varphi_{t,u} : M \to M$ is a C^1-diffeomorphism with inverse $\varphi_{-t,\Theta_t u}$.

Proof:
Let $(t_*,x_*,u_*) \in \mathbb{R} \times M \times \mathcal{U}$ be chosen arbitrarily. Without loss of generality we may assume that $t_* > 0$. We want to show that the derivative of $\varphi_{t_*,u_*} : M \to M$ exists in a neighborhood of x_* and is continuous, which implies that φ_{t_*,u_*} is a C^1-map. Since t_* and u_* are chosen arbitrarily, the assertion then

follows from Corollary 1.2.12. We choose times $0 = \tau_0 < \tau_1 < \cdots < \tau_n = t_*$ and charts $(\phi_1, V_1), \ldots, (\phi_n, V_n)$ of M such that

$$\varphi([\tau_j, \tau_{j+1}], x_*, u_*) \subset V_{j+1} \quad \text{for } j = 0, 1, \ldots, n-1.$$

The map φ_{t_*,u_*} can be written as

$$\varphi_{t_*,u_*} = \varphi_{\tau_n - \tau_{n-1}, u_{n-1}} \circ \cdots \circ \varphi_{\tau_2 - \tau_1, u_1} \circ \varphi_{\tau_1 - \tau_0, u_0}$$

with $u_j = \Theta_{\tau_j} u_*$, $j = 0, 1, \ldots, n-1$, which follows from the cocycle property (1.13) via induction. Hence, it suffices to show that for $j = 0, 1, \ldots, n-1$ the map $\varphi_{\tau_{j+1} - \tau_j, u_j}$ is continuously differentiable in a neighborhood of the point $x_j := \varphi(\tau_j, x_*, u_*)$. To this end, consider the local version of system (1.7) for the control function u_j with respect to the chart (ϕ_{j+1}, V_{j+1}):

$$\dot{y}(t) = \tilde{F}(y(t), u_j(t)), \quad \tilde{F}(y, u) = D\phi_{j+1}(\phi_{j+1}^{-1}(y))F(\phi_{j+1}^{-1}(y), u). \tag{1.18}$$

Let the solutions of (1.18) be denoted by $\tilde{\varphi}(t, y, u_j)$. We have $\varphi([\tau_j, \tau_{j+1}], x_*, u_*) \subset V_{j+1}$ and hence

$$\tilde{\varphi}(t, \phi_{j+1}(x_j), u_j) = \phi_{j+1}(\varphi(t + \tau_j, x_*, u_*)) \quad \text{for all} \quad t \in [0, \tau_{j+1} - \tau_j].$$

By continuous dependence on the initial value (see Corollary 1.2.11) there is a neighborhood $W \subset M$ of x_j such that $\varphi([0, \tau_{j+1} - \tau_j], W, u_j) \subset V_{j+1}$. Hence, we obtain a commutative diagram:

$$\begin{array}{ccc}
W & \xrightarrow{\varphi_{\tau_{j+1}-\tau_j, u_j}} & V_{j+1} \\
{\scriptstyle \phi_{j+1}} \downarrow & & \downarrow {\scriptstyle \phi_{j+1}} \\
\phi_{j+1}(W) & \xrightarrow[\tilde{\varphi}_{\tau_{j+1}-\tau_j, u_j}]{} & \phi_{j+1}(V_{j+1})
\end{array}$$

By Lemma 1.2.13 the map $\tilde{\varphi}_{\tau_{j+1}-\tau_j, u_j}$ is of class C^1 on $\phi_{j+1}(W)$ and hence also $\varphi_{\tau_{j+1}-\tau_j, u_j}$ is continuously differentiable on W. $\square$

The Liouville Formula

We prove the *Liouville Formula* for solutions of control systems (cf. Mane [37, Theorem 3.2, p. 34]). This formula describes the evolution of a volume element in the state space under the "flow" of the control system. Since we consider systems on abstract differentiable manifolds, we need an additional structure to define a volume, namely a volume form. In Section A.1 of the appendix, one finds, amongst others, the definition of volume forms and most of the results used in this subsection.

For the proof of the Liouville Formula in its general form we use the following proposition, which can be regarded as a simpler version. Its proof is essentially copied from Aulbach [4, Satz 6.1.6, pp. 217–218].

1.2.15 Proposition:
Let D be an open subset of $\mathbb{R}^d$ and $f : \mathbb{R} \times D \to \mathbb{R}^d$ a Carathéodory function satisfying the assumptions of Proposition 1.1.11. Denote the maximal solution of the initial value problem

$$\dot{x}(t) = f(t, x), \quad x(0) = x_0,$$

by $t \mapsto \varphi(t, x_0) = \varphi_t(x_0)$, $I(x_0) \to \mathbb{R}^d$. Then for every $x \in D$ the function

$$t \mapsto \det D\varphi_t(x), \quad I(x) \to \mathbb{R},$$

is locally absolutely continuous and satisfies the linear scalar Carathéodory differential equation

$$\dot{z}(t) = \operatorname{tr} D_x f(t, \varphi(t, x)) z(t). \tag{1.19}$$

Proof:
By Proposition 1.1.11 the derivative $D\varphi_t(x) \in \mathbb{R}^{d \times d}$ exists and depends continuously on x and locally absolutely continuous on t. Then also $t \mapsto \det D\varphi_t(x)$ is locally absolutely continuous, since the determinant $\det : \mathbb{R}^{d \times d} \to \mathbb{R}$ is a C^∞-function and hence locally Lipschitz continuous. The function

$$g : \mathbb{R} \times D \to \mathbb{R}, \quad (t, x) \mapsto \operatorname{tr} D_x f(t, \varphi(t, x)),$$

is a Carathéodory function, which follows from the facts that the partial derivatives $\frac{\partial f}{\partial x_i}$, $i = 1, \ldots, d$, exist and are Carathéodory functions (by assumption), that $\varphi(t, x)$ depends continuously on x, and that the trace function is continuous. Hence, (1.19) is a Carathéodory differential equation.

Now, for fixed $x \in D$ denote the entries of the Jacobian $D\varphi_t(x)$ by $w_{ij}(t)$, $i, j = 1, \ldots, d$. Denote the rows of $D\varphi_t(x)$ by $\mu_1(t), \ldots, \mu_d(t)$. Moreover, write $a_{ij}(t)$ for the entries of the matrix $D_x f(t, \varphi(t, x)) \in \mathbb{R}^{d \times d}$. Then, by Proposition 1.1.11, the identity

$$\begin{array}{rcl} \dot{\mu}_1(t) & = & a_{11}(t)\mu_1(t) \; + \; \cdots \; + \; a_{1d}(t)\mu_d(t), \\ \vdots & & \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ \dot{\mu}_d(t) & = & a_{d1}(t)\mu_1(t) \; + \; \cdots \; + \; a_{dd}(t)\mu_d(t) \end{array} \tag{1.20}$$

holds for almost all $t \in I(x)$. The Leibniz Formula for computation of determinants yields

$$\det D\varphi_t(x) = \sum_{\sigma \in \Sigma_d} \operatorname{sign}(\sigma) w_{1\sigma(1)}(t) \cdots w_{d\sigma(d)}(t).$$

Hence, for almost all $t \in I(x)$ we obtain

$$\begin{aligned} \frac{d}{dt} \det D\varphi_t(x) &= \sum_{\sigma \in \Sigma_d} \operatorname{sign}(\sigma) \dot{w}_{1\sigma(1)}(t) \cdots w_{d\sigma(d)}(t) + \cdots \\ &+ \sum_{\sigma \in \Sigma_d} \operatorname{sign}(\sigma) w_{1\sigma(1)}(t) \cdots \dot{w}_{d\sigma(d)}(t). \end{aligned}$$

This implies

$$\frac{d}{dt}\det D\varphi_t(x) = \det\begin{pmatrix}\dot\mu_1(t)\\ \mu_2(t)\\ \vdots\\ \mu_d(t)\end{pmatrix} + \det\begin{pmatrix}\mu_1(t)\\ \dot\mu_2(t)\\ \vdots\\ \mu_d(t)\end{pmatrix} + \cdots + \det\begin{pmatrix}\mu_1(t)\\ \mu_2(t)\\ \vdots\\ \dot\mu_d(t)\end{pmatrix}.$$

From the identity (1.20) and the fact that the determinant of a matrix with two equal rows vanishes it follows that

$$\frac{d}{dt}\det D\varphi_t(x) = \sum_{i=1}^{d} a_{ii}(t)\det\begin{pmatrix}\mu_1(t)\\ \mu_2(t)\\ \vdots\\ \mu_d(t)\end{pmatrix} = \operatorname{tr} D_x f(t,\varphi(t,x))\det D\varphi_t(x).$$

This proves the assertion. □

1.2.16 Theorem:
Consider control system (1.7) *and assume that all maximal solutions are defined on* $\mathbb{R}$. *Let* $\omega \in \Omega_1^d(M)$ *be a* C^1*-volume form on* M. *Then for all* $(t,x,u) \in \mathbb{R}_0^+ \times M \times \mathcal{U}$ *the Liouville Formula holds:*

$$\det\nolimits_\omega D\varphi_{t,u}(x) = \exp\left(\int_0^t \operatorname{div}_\omega F_{u(s)}(\varphi_{s,u}(x))ds\right). \tag{1.21}$$

Proof:
We fix $(x,u) \in M \times \mathcal{U}$. For brevity we write $X_t = F_{u(t)}$ and $x_t = \varphi_{t,u}(x)$ for all $t \in \mathbb{R}$. First we prove that the following identity holds true:

$$\frac{d}{dt}\varphi_{t,u}^*\omega = \varphi_{t,u}^*(\mathcal{L}_{X_t}\omega) \text{ for almost all } t \in \mathbb{R}. \tag{1.22}$$

It suffices to prove formula (1.22) locally (in $\mathbb{R}^d$).[9] Then we have $\omega = \alpha \cdot \omega_0$ with the standard volume form $\omega_0 = dx^1 \wedge \cdots \wedge dx^d$ and a C^1-function $\alpha : \mathbb{R}^d \to \mathbb{R}$. Let $v_1,\ldots,v_d \in \mathbb{R}^d$ be vectors such that (without loss of generality) $\det(v_1|\cdots|v_d) = 1$.[10] Then for all $t \in \mathbb{R}$ we obtain

$$\begin{aligned}\varphi_{t,u}^*\omega(x)(v_1,\ldots,v_d) &= \alpha(x_t)\det(D\varphi_{t,u}(x)v_1|\cdots|D\varphi_{t,u}(x)v_d)\\ &= \alpha(x_t)\det\left[D\varphi_{t,u}(x)\cdot(v_1|\cdots|v_d)\right]\\ &= \alpha(x_t)\det D\varphi_{t,u}(x)\underbrace{\det(v_1|\cdots|v_d)}_{=1}.\end{aligned}$$

For almost all $t \in \mathbb{R}$ the derivatives $\frac{d}{dt}\varphi_{t,u}(x) = \dot x_t$ and $\frac{d}{dt}D\varphi_{t,u}(x)$ exist (see Lemma 1.2.13 for the latter). For those t-values we have

$$\begin{aligned}\frac{d}{dt}\varphi_{t,u}^*\omega(x)(v_1,\ldots,v_d) &= \frac{d}{dt}\left(\alpha(x_t)\det D\varphi_{t,u}(x)\right)\\ &= \langle\nabla\alpha(x_t),\dot x_t\rangle\det D\varphi_{t,u}(x) + \alpha(x_t)\frac{d}{dt}\det D\varphi_{t,u}(x).\end{aligned}$$

[9]We can use the same trick as in the proof of Theorem 1.2.14 and write $\varphi_{t,u}$ as the composition of functions $\varphi_{\tau_1,u_1},\ldots,\varphi_{\tau_n,u_n}$ for sufficiently small times $\tau_1,\ldots,\tau_n$.

[10]By $(v_1|\cdots|v_d)$ we denote the $d\times d$-matrix with columns $v_1,\ldots,v_d$.

By Proposition 1.2.15 we have

$$\frac{d}{dt}\det D\varphi_{t,u}(x) = \operatorname{tr} DX_t(x_t)\det D\varphi_{t,u}(x).$$

This leads to

$$\begin{aligned}
\frac{d}{dt}\varphi_{t,u}^*\omega(x)(v_1,\ldots,v_d) &= \langle\nabla\alpha(x_t), X_t(x_t)\rangle\det D\varphi_{t,u}(x)\\
&+ \alpha(x_t)\operatorname{tr} DX_t(x_t)\det D\varphi_{t,u}(x)\\
&= \left(\langle\nabla\alpha, X_t\rangle + \alpha\operatorname{tr} DX_t\right)(x_t)\det D\varphi_{t,u}(x).
\end{aligned}$$

For the right-hand side of (1.22) we obtain

$$\begin{aligned}
\varphi_{t,u}^*(\mathcal{L}_{X_t}\omega)(x)(v_1,\ldots,v_d) &\overset{(A.17)}{=} \varphi_{t,u}^*(\operatorname{div}_\omega X_t\cdot\omega)(x)(v_1,\ldots,v_d)\\
&\overset{(A.18)}{=} \varphi_{t,u}^*\left((\alpha\operatorname{div}_{\omega_0} X_t + \langle\nabla\alpha, X_t\rangle)\,\omega_0\right)(x)(v_1,\ldots,v_d)\\
&= \left(\alpha\operatorname{tr} DX_t + \langle\nabla\alpha, X_t\rangle\right)(x_t)\det D\varphi_{t,u}(x).
\end{aligned}$$

This proves (1.22). In order to show the assertion, we have to prove that

$$\ln\det_\omega D\varphi_{t,u}(x) = \int_0^t \operatorname{div}_\omega X_s(x_s)ds \quad \text{for all } t\geq 0. \tag{1.23}$$

Note that the integral on the right-hand side of the equation exists, since the function

$$t\mapsto \operatorname{div}_\omega X_t(x_t) = \operatorname{div}_\omega F_{u(t)}(\varphi_{t,u}(x))$$

is the composition of the measurable function $t\mapsto(\varphi(t,x,u),u(t))$, $\mathbb{R}\to M\times\mathbb{R}^m$, and the continuous function $(p,v)\mapsto\operatorname{div}_\omega F_v(p)$, $M\times\mathbb{R}^m\to\mathbb{R}$, and it is essentially bounded on compact intervals: For almost all $s\in[0,t]$ one has

$$\left|\operatorname{div}_\omega F_{u(s)}(\varphi_{s,u}(x))\right| \leq \max_{(z,v)\in\varphi([0,t],x,u)\times U}\left|\operatorname{div}_\omega F_v(z)\right|.$$

For $t=0$ both sides of equation (1.23) coincide, since $\varphi_{0,u}=\mathrm{id}_M$ and hence $\det_\omega D\varphi_{0,u}(x)\equiv 1$. Therefore, it suffices to show that the derivatives of both sides coincide almost everywhere:

$$\begin{aligned}
\tfrac{d}{dt}\ln\det_\omega D\varphi_{t,u}(x) &= \left(\det_\omega D\varphi_{t,u}(x)\right)^{-1}\frac{d}{dt}\det_\omega D\varphi_{t,u}(x)\\
&\overset{(A.16)}{=} \left(\det_\omega D\varphi_{t,u}(x)\right)^{-1}\frac{d}{dt}\frac{\varphi_{t,u}^*\omega(x)}{\omega(x)}\\
&\overset{(1.22)}{=} \left(\det_\omega D\varphi_{t,u}(x)\right)^{-1}\frac{\varphi_{t,u}^*(\mathcal{L}_{X_t}\omega)(x)}{\omega(x)}\\
&\overset{(A.17)}{=} \left(\det_\omega D\varphi_{t,u}(x)\right)^{-1}\frac{\varphi_{t,u}^*([\operatorname{div}_\omega X_t]\cdot\omega)(x)}{\omega(x)}\\
&\overset{(A.16)}{=} \frac{\omega(x)}{(\varphi_{t,u}^*\omega)(x)}\frac{\varphi_{t,u}^*([\operatorname{div}_\omega X_t]\cdot\omega)(x)}{\omega(x)}\\
&= \frac{\varphi_{t,u}^*([\operatorname{div}_\omega X_t]\cdot\omega)(x)}{(\varphi_{t,u}^*\omega)(x)}\\
&= \frac{\operatorname{div}_\omega X_t(x_t)}{\omega(x_t)}\omega(x_t) = \operatorname{div}_\omega X_t(x_t).
\end{aligned}$$

This implies the assertion. □

The Wazewski Inequality

Next, we prove the *Wazewski Inequality* (see, e.g., Boichenko & Leonov & Reitmann [8, p. 42]) for solutions of control systems, which follows from the Riemannian variational equation (see the following proposition). This inequality will allow us to estimate the maximal possible growth of the distance between two solutions for the same control function but different initial values. The distance on the manifold M is supposed to be induced by a Riemannian metric. The facts on Riemannian manifolds used in this subsection can be found in Section A.1 of the appendix.

1.2.17 Proposition:
Consider the control system (1.7) *and assume that all maximal solutions are defined on* $\mathbb{R}$. *Let* g *be a Riemannian metric on* M *of class* C^∞. *For arbitrary* $(x,u) \in M \times \mathcal{U}$ *and* $v \in T_xM$ *the curve*

$$c_{x,u,v} : t \mapsto D\varphi_{t,u}(x)v, \quad c_{x,u,v} : \mathbb{R} \to TM,$$

is locally absolutely continuous and satisfies the variational equation

$$\frac{Dz}{dt}(t) = \nabla F_{u(t)}(\varphi_{t,u}(x))z(t) \tag{1.24}$$

almost everywhere, where $\frac{D}{dt}$ *denotes the covariant derivative along the solution* $\varphi(\cdot,x,u)$.

Proof:
We consider $(x,u) \in M \times \mathcal{U}$ and $v \in T_xM$ as fixed, and we abbreviate $c_{x,u,v}$ by c and $\varphi_{t,u}(x)$ by x_t. Let the local expressions of $F_{u(t)}$ and x_t, respectively, with respect to a chart (ϕ, V) be

$$\begin{aligned} F_{u(t)}(x) &= \sum_i \tilde{F}^i(t,\phi(x))\partial_i\phi_x, \quad \tilde{F}(t,y) :\equiv (\tilde{F}^1(t,y),\ldots,\tilde{F}^d(t,y)), \\ \tilde{x}_t &= \phi(x_t). \end{aligned}$$

By Lemma A.3.6 the local expression of the endomorphism $\nabla F_{u(t)}(x) : T_xM \to T_xM$ is given by

$$w \mapsto \sum_{i,j} \frac{\partial \tilde{F}^i}{\partial y_j}(t,\phi(x))w^j\partial_i\phi_x + \sum_{i,j,k} \Gamma^k_{ij}(x)\tilde{F}^i(t,\phi(x))w^j\partial_k\phi_x$$

for a tangent vector $w = \sum_j w^j \partial_j \phi_x$. In local coordinates let

$$c(t) = \sum_i \tilde{c}^i(t)\partial_i\phi_{x_t}, \quad \tilde{c}(t) :\equiv (\tilde{c}^1(t),\ldots,\tilde{c}^d(t)).$$

Then from Lemma 1.2.13 we know that $\tilde{c}$ (and hence c) is locally absolutely continuous with

$$\dot{\tilde{c}}^i(t) = \sum_j \frac{\partial \tilde{F}^i}{\partial y_j}(t,\tilde{x}_t)\tilde{c}^j(t) \quad \text{almost everywhere}$$

for $i = 1, \ldots, d$. Hence, the right-hand side of (1.24) is (almost everywhere) locally given by

$$\sum_{i,j} \frac{\partial \tilde{F}^i}{\partial y_j}(t, \tilde{x}_t)\tilde{c}^j(t)\partial_i \phi_{x_t} + \sum_{i,j,k} \Gamma^k_{ij}(x_t)\tilde{F}^i(t, \tilde{x}_t)\tilde{c}^j(t)\partial_k \phi_{x_t}$$
$$= \dot{c}(t) + \sum_{i,j,k} \Gamma^k_{ij}(x_t)\dot{\tilde{x}}^i(t)\tilde{c}^j(t)\partial_k \phi_{x_t}.$$

For the left-hand side we obtain

$$\begin{aligned}
\frac{Dc}{dt}(t) &= \frac{D}{dt}\left[\sum_j \tilde{c}^j(t)\partial_j \phi_{x_t}\right] = \sum_j \left[\dot{\tilde{c}}^j(t)\partial_j \phi_{x_t} + \tilde{c}^j(t)\frac{D\partial_j \phi_{x_t}}{dt}(t)\right] \\
&= \dot{c}(t) + \sum_j \tilde{c}^j(t)\left(\nabla_{\dot{x}_t}\partial_j \phi\right)(x_t) \\
&= \dot{c}(t) + \sum_j \tilde{c}^j(t)\left(\nabla_{\sum_i \dot{\tilde{x}}^i(t)\partial_i \phi_{x_t}}\partial_j \phi\right)(x_t) \\
&= \dot{c}(t) + \sum_{i,j} \dot{\tilde{x}}^i(t)\tilde{c}^j(t)\left(\nabla_{\partial_i \phi_{x_t}}\partial_j \phi\right)(x_t) \\
&= \dot{c}(t) + \sum_{i,j,k} \Gamma^k_{ij}(x_t)\dot{\tilde{x}}^i(t)\tilde{c}^j(t)\partial_k \phi_{x_t}.
\end{aligned}$$

This proves the claim. □

1.2.18 Theorem:
Under the assumptions of Proposition 1.2.17 we have

$$\|D\varphi_{t,u}(x)\| \leq \exp\left(\int_0^t \lambda_{\max}(S\nabla F_{u(s)}(\varphi_{s,u}(x)))ds\right) \quad \text{for all } t \geq 0,$$

where $\lambda_{\max}(\cdot)$ denotes the maximal eigenvalue and $S\nabla\cdot$ the symmetrized covariant derivative of a vector field.[11]

Proof:
Let $x_t :\equiv \varphi_{t,u}(x)$ and $\lambda(t) :\equiv \lambda_{\max}(S\nabla F_{u(t)}(\varphi_{t,u}(x)))$. Let $z : \mathbb{R} \to TM$ be a locally absolutely continuous solution of the variational equation (1.24). Then for almost all $t \in \mathbb{R}$ we obtain[12]

$$\begin{aligned}
\frac{d}{dt}\|z(t)\|^2 &= \frac{d}{dt}g_{x_t}(z(t), z(t)) \overset{\text{(A.13)}}{=} g_{x_t}\left(\frac{Dz}{dt}(t), z(t)\right) + g_{x_t}\left(z(t), \frac{Dz}{dt}(t)\right) \\
&= g_{x_t}\left(\nabla F_{u(t)}(x_t)z(t), z(t)\right) + g_{x_t}\left(z(t), \nabla F_{u(t)}(x_t)z(t)\right) \\
&= g_{x_t}\left(\nabla F_{u(t)}(x_t)z(t), z(t)\right) + g_{x_t}\left(\nabla F_{u(t)}(x_t)^* z(t), z(t)\right) \\
&= 2g_{x_t}\left(\tfrac{1}{2}\left[\nabla F_{u(t)}(x_t) + \nabla F_{u(t)}(x_t)^*\right] z(t), z(t)\right) \\
&\leq 2\lambda(t)\|z(t)\|^2.
\end{aligned}$$

[11] The symmetrized covariant derivative is given by $S\nabla X(p) = \frac{1}{2}[\nabla X(p) + \nabla X(p)^*]$ for $X \in \mathcal{X}^1(M)$, $p \in M$.

[12] Note that Formula (A.13) also holds for locally absolutely continuous curves and vector fields, which can be proved by an easy calculation in local coordinates.

Now we assume that $z(t) \neq 0$ for all $t \geq 0$. This implies for almost all $t \geq 0$

$$\begin{aligned}
\frac{\frac{d}{dt}\|z(t)\|^2}{\|z(t)\|^2} \leq 2\lambda(t) \quad &\Rightarrow \quad \int_0^t \frac{\frac{d}{ds}\|z(s)\|^2}{\|z(s)\|^2} ds \leq 2\int_0^t \lambda(s)ds \\
&\Rightarrow \quad \ln\left(\|z(t)\|^2\right) - \ln\left(\|z(0)\|^2\right) \leq 2\int_0^t \lambda(s)ds \\
&\Rightarrow \quad \ln\|z(t)\| - \ln\|z(0)\| \leq \int_0^t \lambda(s)ds \\
&\Rightarrow \quad \|z(t)\| \leq \|z(0)\| \exp\left(\int_0^t \lambda(s)ds\right).
\end{aligned}$$

In order to show that λ is locally integrable (and hence the integral above exists) let (ϕ, V) be a chart such that $\varphi(I, x, u) \subset V$ for some open interval I. Then $\lambda = \lambda_{\max} \circ A$ on I, where $\lambda_{\max}$ is the function assigning to each real symmetric $d \times d$-matrix its maximal eigenvalue, and $A : I \to \mathrm{Sym}(d, \mathbb{R})$ is given by

$$A(t) = SD\tilde{F}_{u(t)}(x_t) + \left[\sum_{i,l} \tilde{F}^i_{u(t)}(x_t) g^{\mu l}(x_t) \frac{\partial g_{\nu l}}{\partial x_i}(x_t)\right]_{\mu,\nu}.$$

Here $\tilde{F}$ is the local expression of F (see also Lemma A.3.6). The function $\lambda_{\max}$ is continuous, since eigenvalues depend continuously on the matrix (see, e.g., Sontag [50, Section A.4]). A is measurable, since both $\tilde{F}_{u(t)}(x_t)$ and $D\tilde{F}_{u(t)}(x_t)$ depend measurably on t, which follows from the facts that $\tilde{F}$ is continuously differentiable, x_t is continuous and u is measurable. Finiteness of the integral (over compact time intervals) follows from compactness of the control range U.

Since for each $v \in T_xM \backslash \{0\}$ the function $z(t) = D\varphi_{t,u}(x)v$ is a solution of (1.24) with $z(t) \neq 0$ for all $t \geq 0$, we obtain

$$\begin{aligned}
\|D\varphi_{t,u}(x)\| &= \max_{\|v\|=1} \|D\varphi_{t,u}(x)v\| \\
&\leq \max_{\|v\|=1} \|\underbrace{D\varphi_{0,u}(x)}_{=\mathrm{id}} v\| \exp\left(\int_0^t \lambda(s)ds\right) = \exp\left(\int_0^t \lambda(s)ds\right),
\end{aligned}$$

which finishes the proof. □

Linearization Along Controlled Trajectories

In this subsection, we introduce the linearization of a control system along a controlled trajectory and show that the solutions of the linearization approximate the solutions of the given system in a neighborhood of the controlled trajectory. Moreover, we discuss controllability of the linearization along periodic trajectories.

1.2.19 Definition:
Consider control system (1.7) *and let* g *be a Riemannian metric on* M *of class*

C^∞. Let $\varphi(\cdot, x_0, u_0) : \mathbb{R} \to M$ be a solution corresponding to an initial value $x_0 \in M$ and a control function $u_0 \in \mathcal{U}$. Define

$$\begin{aligned} A(t) &:= \nabla F_{u_0(t)}(\varphi_{t,u_0}(x_0)) : T_{\varphi_{t,u_0}(x_0)}M \to T_{\varphi_{t,u_0}(x_0)}M, \\ B(t) &:= D_2 F(\varphi_{t,u_0}(x_0), u_0(t)) : \mathbb{R}^m \to T_{\varphi_{t,u_0}(x_0)}M \end{aligned}$$

for all $t \in \mathbb{R}$. The pair $(\varphi(\cdot, x_0, u_0), u_0)$ is called a **controlled trajectory** *and the family*

$$\frac{Dz}{dt}(t) = A(t)z(t) + B(t)\mu(t), \quad \mu \in L^\infty(\mathbb{R}, \mathbb{R}^m), \tag{1.25}$$

of differential equations, where $\frac{D}{dt}$ denotes the covariant derivative along the solution $\varphi(\cdot, x_0, u_0)$, is called the **linearization** *of (1.7) along $(\varphi(\cdot, x_0, u_0), u_0)$. A* **solution** *of (1.25) with initial value $\lambda \in T_{x_0}M$ corresponding to a control function $\mu \in L^\infty(\mathbb{R}, \mathbb{R}^m)$ is a locally absolutely continuous vector field $z : I \to TM$ along $\varphi(\cdot, x_0, u_0)$, defined on an interval I with $0 \in I$, satisfying equation (1.25) for almost all $t \in I$ such that $z(0) = \lambda$.*

1.2.20 Remarks:

- Note that there exist other definitions for the linearization along controlled trajectories. See, e.g., Bullo & Lewis [11], where the linearization of a control-affine system $\dot{x} = f_0(x) + \sum_{i=1}^m u_i f_i(x)$ along a controlled trajectory is defined using the tangent and vertical lifts (with respect to a connection on M) of the vector fields f_i.
- In the following, we will consider control system (1.7) not only with control functions taking values in a fixed compact set U, but with arbitrary L^∞-control functions. It is easy to see that existence and uniqueness of solutions is also guaranteed for this bigger class of control functions.

In the proof of the next proposition, we use some of the statements of Sontag [50, Theorem 1, p. 56]. For convenience of the reader we first formulate a corresponding reduced version of this theorem:

1.2.21 Theorem:
Let M be an open subset of $\mathbb{R}^d$ and $F : M \times \mathbb{R}^m \to \mathbb{R}^d$ a C^1-mapping. Consider the control system

$$\dot{x}(t) = F(x(t), u(t)), \quad u \in L^\infty(\mathbb{R}, \mathbb{R}^m),$$

and denote its solutions by $\varphi(t, x, u)$. For fixed $\tau > 0$ define

$$\mathcal{D}_\tau := \left\{(x, u) \in M \times L^\infty([0, \tau], \mathbb{R}^m) \mid \tau \in I_{\max}(u, x)\right\}.$$

Then $\mathcal{D}_\tau$ is open in $M \times L^\infty([0, \tau], \mathbb{R}^m)$ and the mapping

$$\varphi_\tau : \mathcal{D}_\tau \to M, \quad (x, u) \mapsto \varphi(\tau, x, u),$$

is of class C^1. For fixed $(x_0, u_0) \in M \times L^\infty([0, \tau], \mathbb{R}^m)$ and $(\lambda, \mu) \in \mathbb{R}^d \times L^\infty([0, \tau], \mathbb{R}^m)$ the function

$$\xi(t) := D\varphi_\tau(x_0, u_0)(\lambda, \mu), \quad \xi : [0, \tau] \to \mathbb{R}^d,$$

is a solution of the Carathéodory differential equation

$$\dot{\xi}(t) = D_1F(\varphi(t, x_0, u_0), u_0(t))\xi(t) + D_2F(\varphi(t, x_0, u_0), u_0(t))\mu(t)$$

with initial value λ.[13]

1.2.22 Proposition:
Consider control system (1.7) *assuming that all maximal solutions for arbitrary* L^∞*-control functions are defined on* $\mathbb{R}$. *Let* $(\varphi(\cdot, x_0, u_0), u_0)$ *be a controlled trajectory with corresponding linearization* (1.25). *Then the following statements hold:*

(i) *For every* $\tau > 0$ *the mapping*

$$\varphi_\tau : M \times L^\infty([0, \tau], \mathbb{R}^m) \to M, \quad (x, u) \mapsto \varphi(\tau, x, u),$$

is continuously differentiable.

(ii) *For every initial value* $\lambda \in T_{x_0}M$ *and every control function* $\mu \in L^\infty(\mathbb{R}, \mathbb{R}^m)$ *there exists a unique solution* $\varphi^l(\cdot, \lambda, \mu) : \mathbb{R} \to TM$ *of* (1.25) *satisfying*

$$\varphi^l(t, \lambda, \mu) = D\varphi_t(x_0, u_0)(\lambda, \mu) \tag{1.26}$$

for all $t \in \mathbb{R}$ *and* $(\lambda, \mu) \in T_{x_0}M \times L^\infty(\mathbb{R}, \mathbb{R}^m)$.

(iii) *For every* $\tau > 0$ *the mapping*

$$\varphi^l(\tau, \cdot, \cdot) : T_{x_0}M \times L^\infty([0, \tau], \mathbb{R}^m) \to T_{\varphi(\tau, x_0, u_0)}M$$

is linear and continuous.

(iv) *Assume that the controlled trajectory* $(\varphi(\cdot, x_0, u_0), u_0)$ *is* T_0*-periodic for some* $T_0 > 0$. *Then for all* $k \in \mathbb{Z}$, $t \in \mathbb{R}$ *and* $\lambda \in T_{x_0}M$ *it holds that*

$$\varphi^l(t, \varphi^l(kT_0, \lambda, 0), 0) = \varphi^l(t + kT_0, \lambda, 0).$$

Proof:

(i) Let $(x_0, u_0) \in M \times L^\infty(\mathbb{R}, \mathbb{R}^m)$ be chosen arbitrarily and let (ϕ, V) be a chart around x_0. Define

$$\tilde{F} : \phi(V) \times \mathbb{R}^m \to \mathbb{R}^d, \quad \tilde{F}(y, u) := D\phi_{\phi^{-1}(y)}F(\phi^{-1}(y), u).$$

Then $\tilde{F}$ is a C^1-mapping and we can consider the control system

$$\dot{y}(t) = \tilde{F}(y(t), u(t)), \quad u \in L^\infty(\mathbb{R}, \mathbb{R}^m), \tag{1.27}$$

on the state space $\phi(V) \subset \mathbb{R}^d$. We denote the corresponding solutions by $\tilde{\varphi}(t, y, u)$. By multiplying the right-hand side with a cut-off-function we can extend system (1.27) to $\mathbb{R}^d$ and hence may assume that the maximal

[13] Here $D\varphi_\tau(x_0, u_0)$ denotes the Fréchet differential of φ_τ at (x_0, u_0).

solutions are defined globally. Locally the solutions of (1.27) are related to the solutions of (1.7) by

$$\phi(\varphi(t,x,u)) = \tilde{\varphi}(t,\phi(x),u) \quad \text{for all } (x,u) \in V \times L^\infty(\mathbb{R},\mathbb{R}^m) \tag{1.28}$$

and $t \in \mathbb{R}$ with $\varphi(t,x,u) \in V$. Now, choose $\tilde{\tau} > 0$ such that

$$\tilde{\varphi}([0,\tilde{\tau}],\phi(x_0),u_0) \subset \phi(V).$$

From Theorem 1.2.21 it follows that $\tilde{\varphi}_{\tilde{\tau}}$ is continuous and consequently there exist open neighborhoods $W_1 \subset \phi(V)$ of $\phi(x_0)$ and $W_2 \subset L^\infty([0,\tilde{\tau}],\mathbb{R}^m)$ of u_0 (regarded as an element of $L^\infty([0,\tilde{\tau}],\mathbb{R}^m)$) such that

$$\tilde{\varphi}([0,\tilde{\tau}],y,u) \subset \phi(V) \quad \text{for all } (y,u) \in W_1 \times W_2.$$

Moreover, by Theorem 1.2.21, the mapping

$$\tilde{\varphi}_{\tilde{\tau}} : W_1 \times W_2 \to \phi(V)$$

is of class C^1. Hence, $\varphi_{\tilde{\tau}}$ is of class C^1 on $\phi^{-1}(W_1) \times W_2$.
Now let $\tau > 0$ be arbitrary. We want to show that φ_τ is of class C^1 on an open neighborhood of (x_0,u_0), which proves the assertion. By compactness of $\varphi([0,\tau],x_0,u_0)$ we can find charts $(\phi_1,V_1),\ldots,(\phi_n,V_n)$ of M and times $0 = \tau_0 < \tau_1 < \cdots < \tau_n = \tau$ such that

$$\varphi\left([\tau_{i-1},\tau_i],x_0,u_0\right) \subset V_i \quad \text{for} \quad i = 1,\ldots,n.$$

Moreover, by what we have shown, for $i = 0,1,\ldots,n-1$ we can find open neighborhoods

$$N_i = W_i^1 \times W_i^2 \subset V_{i+1} \times L^\infty([0,\tau_{i+1}-\tau_i],\mathbb{R}^m)$$

of $(x_i,u_i) := (\varphi(\tau_i,x_0,u_0),(\Theta_{\tau_i}u_0)|_{[0,\tau_{i+1}-\tau_i]})$ such that

$$\varphi_{\tau_{i+1}-\tau_i}(N_i) \subset W_{i+1}^1$$

and such that $\varphi_{\tau_{i+1}-\tau_i}$ is of class C^1 on N_i. Define

$$N := W_0^1 \times \bigcap_{i=0}^{n-1} \left\{u \in L^\infty([0,\tau],\mathbb{R}^m) \;:\; \Theta_{\tau_i}\left(u|_{[\tau_i,\tau_{i+1}]}\right) \in W_i^2\right\}.$$

Since the restriction $L^\infty([0,\tau],\mathbb{R}^m) \to L^\infty([\tau_i,\tau_{i+1}],\mathbb{R}^m)$ is continuous, N is an open neighborhood of (x_0,u_0). By the cocycle property (1.13) we have

$$\varphi_{\tau,u} = \varphi_{\tau_n-\tau_{n-1},\Theta_{\tau_{n-1}-\tau_{n-2}}u} \circ \cdots \circ \varphi_{\tau_2-\tau_1,\Theta_{\tau_1-\tau_0}u} \circ \varphi_{\tau_1-\tau_0,u}$$

and hence

$$\varphi_\tau = \varphi_{\tau_n-\tau_{n-1}} \circ \cdots \circ (\varphi_{\tau_2-\tau_1},\Theta_{\tau_1-\tau_0}) \circ \left(\varphi_{\tau_1-\tau_0},\mathrm{id}_{L^\infty([0,\tau],\mathbb{R}^m)}\right).$$

Since the restrictions and shifts of L^∞-functions are linear and continuous, this shows that $\varphi_\tau|_N$ is of class C^1.

(ii) Consider again the local control system (1.27). By Theorem 1.2.21 the function

$$\xi(t) := D\tilde{\varphi}_t(\phi(x_0), u_0)(v, \mu), \quad \xi : I \to \mathbb{R}^d,$$

satisfies the variational equation

$$\dot{\xi}(t) = D_1\tilde{F}(\tilde{\varphi}_{t,u_0}(\phi(x_0)), u_0(t))\xi(t) + D_2\tilde{F}(\tilde{\varphi}_{t,u_0}(\phi(x_0)), u_0(t))\mu(t) \quad (1.29)$$

with initial value $\xi(0) = v = (v^1, \ldots, v^d)$. Here I denotes the maximal interval of definition. In the following, we write

$$x_t := \varphi(t, x_0, u_0) \quad \text{for all } t \in \mathbb{R}.$$

Define

$$\lambda := \sum_i v^i \partial_i \phi_{x_0} \in T_{x_0}M$$

and the vector field

$$z(t) := D\varphi_t(x_0, u_0)(\lambda, \mu), \quad z : \mathbb{R} \to TM,$$

along the trajectory x_t. Let

$$z(t) \equiv \sum_i z^i(t)\partial_i\phi_{x_t}, \quad z^i : I \to \mathbb{R}.$$

By (1.28) we obtain

$$\begin{aligned} z(t) &= D\varphi_t(x_0, u_0)(\lambda, \mu) = D\left(\phi^{-1} \circ \tilde{\varphi}_t \circ (\phi \times \mathrm{id})\right)(x_0, u_0)(\lambda, \mu) \\ &= D\phi^{-1}_{\tilde{\varphi}_t(\phi(x_0), u_0)} D\tilde{\varphi}_t(\phi(x_0), u_0)(\underbrace{D\phi_{x_0}\lambda}_{=v}, \mu) \\ &= D\phi^{-1}_{\phi(\varphi_t(x_0, u_0))}\xi(t) = D\phi^{-1}_{\phi(x_t)}\xi(t). \end{aligned}$$

This proves that $z^i(t) = \xi^i(t)$ (the i^{th} component of $\xi(t)$ with respect to the standard basis). The covariant derivative of z along x_t is given by

$$\frac{Dz}{dt}(t) = \frac{D\left(\sum_i z^i(t)\partial_i\phi_{x_t}\right)}{dt}(t) = \sum_i \left[\dot{z}^i(t)\partial_i\phi_{x_t} + z^i(t)\nabla_{\dot{x}_t}\partial_i\phi_{x_t}\right].$$

Let $\dot{x}_t = \sum_j w^j(t)\partial_j\phi_{x_t}$. Then

$$\begin{aligned} \frac{Dz}{dt}(t) &= \sum_i \left[\dot{z}^i(t)\partial_i\phi_{x_t} + z^i(t)\sum_j w^j(t)\nabla_{\partial_j\phi_{x_t}}\partial_i\phi_{x_t}\right] \\ &= \sum_i \left[\dot{z}^i(t)\partial_i\phi_{x_t} + z^i(t)\sum_{j,k}\Gamma^k_{ij}(x_t)w^j(t)\partial_k\phi_{x_t}\right] \\ &= \sum_k \left[\dot{z}^k(t) + \sum_{i,j}\Gamma^k_{ij}(x_t)z^i(t)w^j(t)\right]\partial_k\phi_{x_t}. \end{aligned}$$

Let $F(x,u) = \sum_i F_u^i(x)\partial_i\phi_x$. Then

$$\begin{aligned}\tilde F(y,u) &= D\phi_{\phi^{-1}(y)} F(\phi^{-1}(y),u)\\ &= \sum_i F_u^i(\phi^{-1}(y))\,\underbrace{D\phi_{\phi^{-1}(y)}\partial_i\phi_{\phi^{-1}(y)}}_{=e_i}.\end{aligned}$$

Hence, $F_u^i(\phi^{-1}(y))$ is the i^{th} component of $\tilde F(y,u)$ with respect to the standard basis. The covariant derivative of $F_{u_0(t)}$ at x_t, applied to $z(t)$, is given by

$$\begin{aligned}\nabla_{z(t)}F_{u_0(t)}(x_t) &= \sum_i z^i(t)\nabla_{\partial_i\phi_{x_t}}\left(\sum_j F^j_{u_0(t)}(x_t)\partial_j\phi_{x_t}\right)\\ &= \sum_{i,j} z^i(t)\left[\partial_i\phi_{x_t}\left(F^j_{u_0(t)}\right)\partial_j\phi_{x_t} + F^j_{u_0(t)}(x_t)\nabla_{\partial_i\phi_{x_t}}\partial_j\phi_{x_t}\right]\\ &= \sum_{i,j} z^i(t)\Big[\partial_i\phi_{x_t}\left(F^j_{u_0(t)}\right)\partial_j\phi_{x_t}\\ &\quad + F^j_{u_0(t)}(x_t)\sum_k \Gamma^k_{ij}(x_t)\partial_k\phi(x_t)\Big]\\ &= \sum_k\Big[\sum_i \partial_i\phi_{x_t}\left(F^k_{u_0(t)}\right)z^i(t)\\ &\quad + \sum_{i,j}\Gamma^k_{ij}(x_t)z^i(t)\underbrace{F^j_{u_0(t)}(x_t)}_{=w^j(t)}\Big]\partial_k\phi_{x_t}.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\frac{Dz}{dt}(t) - \nabla_{z(t)}F_{u_0(t)}(x_t) &= \sum_k\left[\dot z^k(t) - \sum_i \partial_i\phi_{x_t}\left(F^k_{u_0(t)}\right)z^i(t)\right]\partial_k\phi_{x_t}\\ &= \sum_k\left[\dot\xi^k(t) - \sum_i \frac{\partial(F^k_{u_0(t)}\circ\phi^{-1})}{\partial y_i}(\phi(x_t))\xi^i(t)\right]\partial_k\phi_{x_t}.\end{aligned}$$

Applying $D\phi_{x_t}$ to this equation gives

$$\begin{aligned}D\phi_{x_t}\left(\frac{Dz}{dt}(t) - \nabla_{z(t)}F_{u_0(t)}(x_t)\right) &= \dot\xi(t) - D_1\tilde F(\phi(x_t),u_0(t))\xi(t)\\ &\overset{(1.29)}{=} D_2\tilde F(\phi(x_t),u_0(t))\mu(t).\end{aligned}$$

By linearity of $D\phi_{x_t}$ this implies

$$\begin{aligned}\frac{Dz}{dt}(t) - \nabla_{z(t)}F_{u_0(t)}(x_t) &= (D\phi_{x_t})^{-1}D_2\tilde F(\phi(x_t),u_0(t))\mu(t)\\ &= (D\phi_{x_t})^{-1}D_2 D\phi_{x_t}F(x_t,u_0(t))\mu(t)\\ &= D_2(D\phi_{x_t})^{-1}D\phi_{x_t}F(x_t,u_0(t))\mu(t)\\ &= D_2F(x_t,u_0(t))\mu(t).\end{aligned}$$

This proves that z (locally) satisfies equation (1.25) with initial condition λ and control μ. By using the cocycle property and an argument similar to that in the proof of (i), one easily shows that this implies (1.26), provided that uniqueness of the solution holds. Since uniqueness is a local issue and equation (1.25) locally reduces to the variational equation in $\mathbb{R}^d$, as we have seen, the assertion holds.

(iii) This immediately follows from statement (ii).

(iv) We write

$$X(t) := \varphi^l(t + kT_0, \lambda, 0), \quad Y(t) := \varphi^l(t, \varphi^l(kT_0, \lambda, 0), 0).$$

Both X and Y are locally absolutely continuous vector fields along $\varphi(\cdot, x_0, u_0)$ and $X(0) = \varphi^l(kT_0, \lambda, 0) = Y(0)$. By periodicity of $(\varphi(\cdot, x_0, u_0), u_0)$ also $A(t)$ is periodic with the same period T_0 and hence

$$\frac{DX}{dt}(t) = A(t + kT_0)X(t) = A(t)X(t), \quad \frac{DY}{dt}(t) = A(t)Y(t).$$

By uniqueness of solutions it follows that $X = Y$.

□

1.2.23 Remark:
Proposition 1.2.22(ii) shows that the linearization (1.25) is an object which actually does <u>not</u> depend on the Riemannian metric g, since the solutions are the same for every metric.

The following proposition shows that the solutions of the linearization (1.25) approximate the solutions of the nonlinear system (1.7) in a neighborhood of the controlled trajectory.

1.2.24 Proposition:
Consider control system (1.7) *and its linearization along the controlled trajectory* $(\varphi(\cdot, x_0, u_0), u_0)$. *Then for all* $\tau, C > 0$ *there exist* $\delta > 0$ *and a function* $\zeta = \zeta_{\tau,C} : [0, \delta) \to \mathbb{R}_0^+$ *with*

$$\lim_{b \searrow 0} \zeta(b) = 0$$

such that

$$\left\| \exp^{-1}_{\varphi(\tau,x_0,u_0)}(\varphi(\tau, x, u)) - \varphi^l(\tau, \exp^{-1}_{x_0}(x), u - u_0) \right\| \leq \zeta(b)b \tag{1.30}$$

for all $x \in M$ *with* $d(x, x_0) \leq b$ *and* $u \in L^\infty([0, \tau], \mathbb{R}^m)$ *with* $\|u - u_0\|_{[0,\tau]} \leq Cb$, *where* $b \in [0, \delta)$ *is small enough that* $\exp^{-1}_{x_0}(x)$ *and* $\exp^{-1}_{\varphi(\tau,x_0,u_0)}(\varphi(\tau, x, u))$ *are defined (i.e.,* x *and* $\varphi(\tau, x, u)$ *are contained in the range of the local diffeomorphisms, defined by restriction of* $\exp_{x_0}$ *and* $\exp_{\varphi(\tau,x_0,u_0)}$ *to appropriate open neighborhoods of* $0 \in T_{x_0}M$ *and* $0 \in T_{\varphi(\tau,x_0,u_0)}M$*, respectively).*

Proof:
For given $\tau > 0$ consider the mappings

$$\alpha : M \times L^\infty([0, \tau], \mathbb{R}^m) \to M, \quad (x, u) \mapsto \varphi(\tau, x, u),$$

and

$$\begin{aligned}\tilde{\alpha} : T_{x_0}M \times L^\infty([0,\tau],\mathbb{R}^m) \supset \widetilde{W} &\to T_{\varphi(\tau,x_0,u_0)}M \\ (y,u) &\mapsto \exp^{-1}_{\varphi(\tau,x_0,u_0)}(\alpha(\exp_{x_0}(y),u)),\end{aligned}$$

where $\widetilde{W}$ is an open neighborhood of $(0,u_0) \in T_{x_0}M \times L^\infty([0,\tau],\mathbb{R}^m)$, chosen small enough such that $\tilde{\alpha}$ is well-defined. Since $\alpha(\exp_{x_0}(0),u_0) = \varphi(\tau,x_0,u_0)$ and α is continuous, which follows from Proposition 1.2.22(i), such $\widetilde{W}$ exists. By Proposition 1.2.22(i) both α and $\tilde{\alpha}$ are continuously differentiable. Differentiating $\tilde{\alpha}$ at $(0,u_0)$ by the chain rule yields

$$D\tilde{\alpha}_{(0,u_0)}(\lambda,\mu) = D\exp^{-1}_{\varphi(\tau,x_0,u_0)}(\varphi(\tau,x_0,u_0))D\alpha_{(x_0,u_0)}D(\exp_{x_0} \times \mathrm{id})(\lambda,\mu).$$

Using that $D\alpha_{(x_0,u_0)}(\lambda,\mu) = \varphi^l(\tau,\lambda,\mu)$ (see Proposition 1.2.22(ii)) and that the derivative of the Riemannian exponential map at 0 is the identity (see (A.14)) we obtain

$$D\tilde{\alpha}_{(0,u_0)}(\lambda,\mu) = \varphi^l(\tau,\lambda,\mu).$$

Thus,

$$\begin{aligned}\exp^{-1}_{\varphi(\tau,x_0,u_0)}(\varphi(\tau,\exp_{x_0}(y),u)) &= \tilde{\alpha}(y,u) \\ &= \underbrace{\tilde{\alpha}(0,u_0)}_{=0} + D\tilde{\alpha}_{(0,u_0)}(y,u-u_0) + r(y,u) \\ &= \varphi^l(\tau,y,u-u_0) + r(y,u)\end{aligned}$$

for all $(y,u) \in \widetilde{W}$, where $r(y,u)$ satisfies

$$\lim_{(y,u)\to(0,u_0)} \frac{r(y,u)}{\|y\| + \|u\|_{[0,\tau]}} = 0. \tag{1.31}$$

Hence, we obtain

$$\left\|\exp^{-1}_{\varphi(\tau,x_0,u_0)}(\varphi(\tau,\exp_{x_0}(y),u)) - \varphi^l(\tau,y,u-u_0)\right\| \equiv \|r(y,u)\|. \tag{1.32}$$

Since $\widetilde{W}$ is an open neighborhood of $(0,u_0)$, for given $C > 0$ there exists $\delta > 0$ such that the $B_\delta(0) \times B_{C\delta}(u_0) \subset \widetilde{W}$. Define $\zeta_{\tau,C} : [0,\delta) \to \mathbb{R}_0^+$ by

$$\zeta_{C,\tau}(b) := \begin{cases} b^{-1} \sup_{\substack{\|y\|\le b,\\ \|u-u_0\|_{[0,\tau]}\le Cb}} \|r(y,u)\| & \text{for } b \in (0,\delta), \\ 0 & \text{for } b = 0. \end{cases}$$

Then from (1.32) we obtain (1.30). From (1.31) it follows that for every $\varepsilon > 0$ there is $b > 0$ such that $\|y\| \le b$ and $\|u-u_0\|_{[0,\tau]} \le Cb$ implies $\frac{\|r(y,u)\|}{\|y\|+\|u\|_{[0,\tau]}} \le \varepsilon$. Hence, from

$$\frac{\|r(y,u)\|}{\|y\| + \|u\|_{[0,\tau]}} = \frac{\|r(y,u)\|}{b}\frac{b}{\|y\| + \|u\|_{[0,\tau]}} \le \varepsilon$$

it follows that

$$\frac{\|r(y,u)\|}{b} \leq \varepsilon \frac{\|y\| + \|u\|_{[0,\tau]}}{b} \leq \varepsilon \frac{b(C+1)}{b} = \varepsilon(C+1).$$

For $b = b(\varepsilon)$ this implies

$$\zeta_{C,\tau}(b) = \sup_{\substack{\|y\|\leq b, \\ \|u-u_0\|_{[0,\tau]} \leq Cb}} \frac{\|r(y,u)\|}{b} \leq \varepsilon(C+1),$$

which finishes the proof. □

Next, we introduce the notion of controllability for the linearization along a periodic controlled trajectory.

1.2.25 Definition:
Let $(\varphi(\cdot, x_0, u_0), u_0)$ be a T_0-periodic controlled trajectory of system (1.7)*. Then the linearization along $(\varphi(\cdot, x_0, u_0), u_0)$ is called* **controllable** *if for all $\lambda_1, \lambda_2 \in T_{x_0}M$ there exists $\mu \in L^\infty([0,T_0], \mathbb{R}^m)$ such that*

$$\varphi^l(T_0, \lambda_1, \mu) = \lambda_2.$$

1.2.26 Proposition:
Under the assumptions of Definition 1.2.25 system (1.25) *is controllable if and only if for each $\lambda \in T_{x_0}M$ there is $\mu \in L^\infty([0,T_0], \mathbb{R}^m)$ with*

$$\varphi^l(T_0, \lambda, \mu) = 0.$$

Proof:
We only show the nontrivial direction: Let $\lambda_1, \lambda_2 \in T_{x_0}M$ and define $\tilde{\lambda}_2 := \varphi^l(-T_0, \lambda_2, 0) \in T_{x_0}M$. Then one finds $\mu \in L^\infty([0,T_0], \mathbb{R}^m)$ with

$$\begin{aligned} 0 &= \varphi^l(T_0, \lambda_1 - \tilde{\lambda}_2, \mu) = \varphi^l(T_0, \lambda_1, \mu) - \varphi^l(T_0, \tilde{\lambda}_2, 0) \\ &= \varphi^l(T_0, \lambda_1, \mu) - \varphi^l(T_0, \varphi^l(-T_0, \lambda_2, 0), 0) \\ &= \varphi^l(T_0, \lambda_1, \mu) - \lambda_2. \end{aligned}$$

Here we used Proposition 1.2.22(iii) and (iv). □

1.2.27 Proposition:
Consider control system (1.7) *and its linearization along the T_0-periodic controlled trajectory $(\varphi(\cdot, x_0, u_0), u_0)$. If the linearization is controllable, then there exists $C > 0$ such that for all $\lambda \in T_{x_0}M$ there is $\mu \in L^\infty([0,T_0], \mathbb{R}^m)$ with*

$$\varphi^l(T_0, \lambda, \mu) = 0 \quad \text{and} \quad \|\mu\|_{[0,T_0]} \leq C\|\lambda\|.$$

Proof:
By controllability, for every $\lambda \in T_{x_0}M$ there exists at least one $\mu \in L^\infty([0,T_0], \mathbb{R}^m)$ such that $\varphi^l(T_0, \lambda, \mu) = 0$ or equivalently

$$\varphi^l(T_0, 0, \mu) = \varphi^l(T_0, -\lambda, 0).$$

Consider the automorphism $Q : T_{x_0}M \to T_{x_0}M$, $Q\lambda := \varphi^l(T_0, -\lambda, 0)$ and the continuous linear operator

$$L : L^\infty([0,T_0],\mathbb{R}^m) \to T_{x_0}M, \quad \mu \mapsto \varphi^l(T_0, 0, \mu).$$

Obviously, controllability is equivalent to L being surjective. Hence, by Bachman & Narici [6, Theorem 16.5, p. 265] (the *bounded inverse theorem*), there exists a constant $\tilde{C} > 0$ such that for all $\lambda \in T_{x_0}M$ there is $\mu \in L^\infty([0,T_0],\mathbb{R}^m)$ with $L\mu = Q\lambda$ and $\|\mu\|_{[0,T_0]} \leq \tilde{C}\|Q\lambda\|$. Thus, with $C := \tilde{C}\|Q\|$ the assertion holds. □

1.2.28 Remark:
For control systems on Euclidean space our notion of controllability for the linearization along a (periodic) controlled trajectory coincides with the usual one, as it is defined, e.g., in Sontag [50, Chapter 3]. In particular, in the case of a controlled equilibrium (x_0, u_0), controllability of the linearization is equivalent to controllability of the matrix pair (A, B) with $A = D_xF(x_0, u_0)$ and $B = D_uF(x_0, u_0)$, which means that the *controllability matrix* $[B|AB|\cdots|A^{d-1}B]$ has full rank (cf. Sontag [50, Section 3.2]).

Approximation by Simple Controls

In this subsection, we use results from Grasse & Sussmann [26] in order to obtain a theorem on the approximation of arbitrary trajectories of a control system by trajectories corresponding to piecewise constant control functions. To this end, we first explain how control systems are defined in [26] and why the results there can be applied to control systems in the sense of our definition.

In [26, Definition 2.2, p. 35] a C^1*-control system* is defined as a mapping

$$f : M \times \Omega \to TM,$$

where M is a connected, d-dimensional, second-countable, Hausdorff, differentiable manifold of class C^k for $k \geq 2$, and Ω is a separable metric space, such that the following properties are fulfilled:

(i) For each $\omega \in \Omega$ the map $f_\omega : M \to TM$, $x \mapsto f(x, \omega)$, is a vector field.

(ii) For every chart (ϕ, V) of M the map $f_\phi : \phi(V) \times \Omega \to \mathbb{R}^d$, given by

$$f_\phi(y, \omega) = D\phi_{\phi^{-1}(y)} f(\phi^{-1}(y), \omega),$$

is of class C^1 in its first variable, i.e., for every y the map $y \mapsto f_\phi(y, \omega)$ is of class C^1 and f_ϕ, $D_1 f_\phi$ are continuous on $\phi(V) \times \Omega$.

Obviously, the right-hand side of a control system in the sense of our definition satisfies these properties and hence we can use the results of [26].

In [26, Definition 2.9, p. 37], for a C^1-control system $f : M \times \Omega \to TM$ a Lebesgue measurable function $u : \mathbb{R} \to \Omega$ is said to be an *admissible control* *function* if for every chart (ϕ, V) of M the map $h(y,t) := f_\phi(y, u(t))$, $h : \phi(V) \times \Omega \to \mathbb{R}^d$, satisfies the following properties:

(i) For every $t \in \mathbb{R}$ the map $y \mapsto h(y,t)$ is of class C^1.

(ii) For every $y \in \mathbb{R}^d$ the maps $t \mapsto h(y,t)$ and $t \mapsto D_1 h(y,t)$ are measurable.

(iii) For every $(y_0, t_0) \in \phi(V) \times \mathbb{R}$ there exists $\delta > 0$ and $\lambda \in L^1([t_0-\delta, t_0+\delta], \mathbb{R})$ such that $\|h(y,t)\| + \|D_1 h(y,t)\| \leq \lambda(t)$ for all $t \in [t_0 - \delta, t_0 + \delta]$.

The set of all admissible control functions in this sense is denoted by $\mathcal{U}_{\mathrm{meas}}(f)$. Since our control systems have a continuously differentiable right-hand side and a compact control range, it is clear that the control functions we consider are elements of $\mathcal{U}_{\mathrm{meas}}(f)$. Indeed, this is proved in [26, Example 2.10, pp. 37–38].

For a C^1-control system $f : M \times \Omega \to TM$ a topology on $\mathcal{U}_{\mathrm{meas}}(f)$, called the *$f$-topology*, is defined via a family of pseudo-metrics (see Definition 2.13 in [26], p. 39). A control function $u \in \mathcal{U}_{\mathrm{meas}}(f)$ is called a *step map* (see Definition 2.3 in [26], p. 35), if there exist $t_1 < t_2 < \cdots < t_p$ in $\mathbb{R}$ such that u is constant on the intervals $(-\infty, t_1)$, (t_1, t_2),...,(t_{p-1}, t_p), (t_p, ∞). The set of all step maps is denoted by $\mathcal{U}_{\mathrm{step}}$.

By [26, Theorem 2.20, p. 45] for a C^1-control system $f : M \times \Omega \to TM$ the set $\mathcal{U}_{\mathrm{step}}$ is a dense subset of $\mathcal{U}_{\mathrm{meas}}(f)$ in the f-topology. The following proposition now immediately follows from the observations above and [26, Theorem 2.24, p. 48]:

1.2.29 Proposition:
Consider control system (1.7) *assuming that all maximal solutions are defined on* $\mathbb{R}$. *Let* d_M *denote a metric on* M *compatible with the given topology. Let* $(x_0, u_0) \in M \times \mathcal{U}$ *and* $T > 0$. *Then for every* $\varepsilon > 0$ *there exist* $\delta > 0$ *and a neighborhood* $\mathcal{N}$ *of* u_0 *in the* F*-topology on* $\mathcal{U}$ *such that* $d_M(x, x_0) < \delta$ *and* $u \in \mathcal{N}$ *imply*

$$d_M\left(\varphi(t,x,u), \varphi(t,x_0,u_0)\right) \leq \varepsilon \quad \text{for every } t \in [0,T].$$

By the fact that $\mathcal{U}_{\mathrm{step}}$ is dense in $\mathcal{U}_{\mathrm{meas}}(F)$ the next corollary immediately follows:

1.2.30 Corollary:
Consider control system (1.7) *assuming that all maximal solutions are defined on* $\mathbb{R}$. *Let* d_M *denote a metric on* M *compatible with the given topology. Let* $(x_0, u_0) \in M \times \mathcal{U}$ *and* $T > 0$. *Then for every* $\varepsilon > 0$ *there exist* $\delta > 0$ *and a piecewise constant control function* $u \in \mathcal{U}$ *such that* $d_M(x, x_0) < \delta$ *implies*

$$d_M\left(\varphi(t,x,u), \varphi(t,x_0,u_0)\right) \leq \varepsilon \quad \text{for every } t \in [0,T].$$

1.3 Qualitative Theory

In this section, we cite some results from Colonius & Kliemann [16] on the qualitative behavior of control systems. In particular, we introduce the important notion of control sets, which are the maximal subsets of the state space on which approximate controllability holds. We consider the control system (1.7) and assume that all solutions are defined globally, i.e., that the cocycle φ is defined on $\mathbb{R} \times M \times \mathcal{U}$.

1.3.1 Definition:
Let $x \in M$. The set of points **reachable from x up to time $T > 0$** *is*

$$\mathcal{O}^+_{\leq T}(x) := \{y \in M \mid \exists t \in [0,T],\ u \in \mathcal{U}:\ y = \varphi(t,x,u)\}\,.$$

The set of points **controllable to x within time $T > 0$** *is*

$$\mathcal{O}^-_{\leq T}(x) := \{y \in M \mid \exists t \in [0,T],\ u \in \mathcal{U}:\ x = \varphi(t,y,u)\}\,.$$

Furthermore, we define

$$\mathcal{O}^+(x) := \bigcup_{T>0} \mathcal{O}^+_{\leq T}(x), \quad \mathcal{O}^-(x) := \bigcup_{T>0} \mathcal{O}^-_{\leq T}(x).$$

We call $\mathcal{O}^+(x)$ and $\mathcal{O}^-(x)$ the **positive and negative orbit of x**, *respectively.*

1.3.2 Definition (Controlled Invariance):
A set $D \subset M$ is called **controlled invariant** *if for every $x \in D$ there exists some $u \in \mathcal{U}$ such that $\varphi(\mathbb{R}^+_0, x, u) \subset D$. D is called* **(positively) invariant** *if $\mathcal{O}^+(x) \subset D$ for all $x \in D$.*

1.3.3 Definition (Control Set):
A set $D \subset M$ is called a **control set** *of system* (1.7) *if it has the following properties:*

(i) *D is controlled invariant.*

(ii) *For all $x \in D$ one has $D \subset \operatorname{cl} \mathcal{O}^+(x)$.*

(iii) *D is maximal with the properties (i) and (ii), i.e., if $D' \supset D$ has the properties (i) and (ii), then $D' = D$.*

If $\operatorname{cl} \mathcal{O}^+(x) = \operatorname{cl} D$ for all $x \in D$, then D is called an **invariant control set**, *otherwise a* **variant control set**.

1.3.4 Definition (Local Accessibility):
The control system (1.7) *is called* **locally accessible from $x \in M$** *if the sets $\operatorname{int} \mathcal{O}^+_{\leq T}(x)$ and $\operatorname{int} \mathcal{O}^-_{\leq T}(x)$ are nonvoid for all $T > 0$. The system is called locally accessible if it is locally accessible from every point $x \in M$.*

The following theorem gives a sufficient condition for local accessibility. The proof can be found in [16, Theorem A.4.4, p. 526].

1.3.5 Theorem (Krener):
Consider the control system (1.7) *and assume that* $F(\cdot, u) : M \to TM$ *is a complete* C^∞*-vector field for all* $u \in U$*. Define*

$$\mathcal{F} := \{F(\cdot, u) \mid u \in U\} \subset \mathcal{X}^\infty(M).$$

Let $\mathcal{L}(\mathcal{F}) \subset \mathcal{X}^\infty(M)$ *be the smallest Lie algebra containing the set* $\mathcal{F}$ *and* $\Delta_{\mathcal{L}(\mathcal{F})}(p) := \{X(p) \mid X \in \mathcal{L}(\mathcal{F})\}$*. Then, if* $\Delta_{\mathcal{L}(\mathcal{F})}(p) = T_pM$ *for all* $p \in M$*, the system is locally accessible.*

The next proposition can be found in [16, Lemma 3.2.13, p. 60].

1.3.6 Proposition:
Let D *be a control set of system* (1.7) *with nonvoid interior. Then the following assertions hold:*

(i) *If the system is locally accessible from all* $x \in \operatorname{cl} D$*, then* D *is connected and* $\operatorname{cl}\operatorname{int} D = \operatorname{cl} D$*.*

(ii) *If the system is locally accessible from* $y \in \operatorname{int} D$*, then* $y \in \mathcal{O}^+(x)$ *for all* $x \in D$*.*

(iii) *If the system is locally accessible from all* $y \in \operatorname{int} D$*, then* $\operatorname{int} D \subset \mathcal{O}^+(x)$ *for all* $x \in D$*, and for every* $y \in \operatorname{int} D$ *one has*

$$D = \operatorname{cl} \mathcal{O}^+(y) \cap \mathcal{O}^-(y). \tag{1.33}$$

Another important property of a control set D with nonvoid interior is that trajectories starting in D cannot leave D and return, which is also called the *no-return property*. The following propositions are taken from [16, Proposition 3.2.3, p. 54] and [16, Proposition 3.2.4, p. 54].

1.3.7 Proposition:
Let $D \subset M$ *be a maximal set with the property that for all* $x \in D$ *one has* $D \subset \operatorname{cl} \mathcal{O}^+(x)$ *and suppose that for some* $x \in D$ *there are* $T > 0$ *and* $u \in \mathcal{U}$ *with* $\varphi(T, x, u) \in D$*. Then* $\varphi(t, x, u) \in D$ *for all* $t \in [0, T]$*.*

1.3.8 Proposition:
Let $D \subset M$ *be a maximal set with the property that for all* $x \in D$ *one has* $D \subset \operatorname{cl} \mathcal{O}^+(x)$ *and suppose that the interior of* D *is nonvoid. Then* D *is a control set.*

1.3.9 Remark:
It easily follows that also the converse of Proposition 1.3.8 is true: Let $D \subset M$ be a control set with nonvoid interior. Assume to the contrary that D is not maximal with $D \subset \operatorname{cl} \mathcal{O}^+(x)$ for all $x \in D$. Then D is a proper subset of a set D', which is maximal with this property. Then, due to the proposition, D' is a control set, which implies the contradiction that D is not a control set.

The boundary of a control set with nonvoid interior allows a partition into three subsets with different dynamical properties. This decomposition is described in the following definition.

1.3.10 Definition:
For a control set D with nonvoid interior we define the following subsets of the boundary:

$$\begin{aligned}
\Gamma(D) &:= \{x \in \operatorname{bd} D \mid \exists y \in \operatorname{int} D: \ u \in \mathcal{U}: \ T > 0 : x = \varphi(T, y, u)\}\,, \\
\Gamma^*(D) &:= \{x \in \operatorname{bd} D \mid \exists y \in \operatorname{int} D: \ u \in \mathcal{U}: \ T > 0 : y = \varphi(T, x, u)\}\,, \\
\tilde{\Gamma}(D) &:= \left\{x \in \operatorname{bd} D \mid \mathcal{O}^+(x) \cap D = \emptyset \text{ and } \mathcal{O}^-(x) \cap D = \emptyset\right\}.
\end{aligned}$$

These sets are called the **exit, entrance** *and* **tangential boundaries** *of D, respectively.*

The next proposition describes topological properties of these sets (see [16, Proposition 3.2.25, p. 66]).

1.3.11 Proposition:
Let D be a control set with nonvoid interior such that local accessibility holds on $\operatorname{cl} D$. Then the following is true:

(i) *The sets $\Gamma(D)$, $\Gamma^*(D)$ and $\tilde{\Gamma}(D)$ form a decomposition of $\operatorname{bd} D$.*

(ii) *The sets $\Gamma(D)$ and $\Gamma^*(D)$ are open in $\operatorname{bd} D$ and $\tilde{\Gamma}(D)$ is closed in $\operatorname{bd} D$.*

(iii) *The equality $\tilde{\Gamma}(D) = \operatorname{cl}\Gamma(D) \cap \operatorname{cl}\Gamma^*(D)$ holds and $\operatorname{int}_{\operatorname{bd} D} \tilde{\Gamma}(D) = \emptyset$.*

1.3.12 Remark:
Although the right-hand sides of control systems are assumed to be of class C^∞ frequently in [16], the results on control sets also hold true for systems with less smoothness, since in the proofs only continuous dependence on initial conditions is used.

Next, we introduce an important class of control systems, namely control-affine systems.

1.3.13 Definition:
Let $F : M \times \mathbb{R}^m \to TM$ be defined by $F(x, u) := f_0(x) + \sum_{i=1}^m u_i f_i(x)$ with $f_0, f_1, \ldots, f_m \in \mathcal{X}^1(M)$, and let $U \subset \mathbb{R}^m$ be compact and convex. Then the corresponding control system

$$\dot{x}(t) = F(x(t), u(t)) = f_0(x(t)) + \sum_{i=1}^m u_i(t) f_i(x(t)), \quad u \in \mathcal{U}, \tag{1.34}$$

is called a **control-affine system**. *f_0 is called the* **drift vector field** *and $f_1, \ldots, f_m$ the* **control vector fields**.

Obviously, everything which is true for the more general systems defined in Definition 1.2.1 also holds for control-affine systems. But moreover, it can be shown that for these systems $\mathcal{U}$ becomes a compact metrizable space when endowed with the weak*-topology of $L^\infty(\mathbb{R}, \mathbb{R}^m) = L^1(\mathbb{R}, \mathbb{R}^m)^*$ and both the shift flow Θ and the cocycle φ are continuous with respect to this topology.[14] See the following proposition. (For the proof see [16, Lemma 4.2.1, p. 95], [16, Lemma 4.2.4, p. 96] and [16, Lemma 4.3.2, p. 100]).

1.3.14 Proposition:
Consider the control-affine system (1.34) *and let* $\mathcal{U}$ *be endowed with the weak***-topology. Then the following assertions hold:*

(i) *$\mathcal{U}$ is a compact, separable metrizable space. A metric is given by*

$$d_{\mathcal{U}}(u,v) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\left| \int_{\mathbb{R}} \langle u(t) - v(t), x_n(t) \rangle dt \right|}{1 + \left| \int_{\mathbb{R}} \langle u(t) - v(t), x_n(t) \rangle dt \right|}, \tag{1.35}$$

where $\{x_n \mid n \in \mathbb{N}\}$ is an arbitrary countable dense subset of $L^1(\mathbb{R},\mathbb{R}^m)$ and $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{R}^m$.

(ii) *The shift flow $\Theta : \mathbb{R} \times \mathcal{U} \to \mathcal{U}$ and the cocycle $\varphi : \mathbb{R} \times M \times \mathcal{U} \to M$ are continuous, and the mapping*

$$\Phi : \mathbb{R} \times M \times \mathcal{U} \to M \times \mathcal{U}, \quad \Phi(t,x,u) := (\varphi(t,x,u), \Theta_t u), \tag{1.36}$$

is a continuous flow, the so-called **control flow** *of system* (1.34).

For each pair $(x,u) \in M \times \mathcal{U}$ the ω-limit set $\omega(x,u) \subset M \times \mathcal{U}$ with respect to the control flow is

$$\omega(x,u) = \left\{ (y,v) \in M \times \mathcal{U} \mid \exists t_k \to \infty : \ (\varphi(t_k,x,u), \Theta_{t_k} u) \to (y,v) \right\}.$$

By $\pi_M : M \times \mathcal{U} \to M$ we denote the projection onto the first factor, i.e., $\pi_M(x,u) = x$. Then

$$\pi_M \omega(x,u) = \left\{ y \in M \mid \exists t_k \to \infty : \ \varphi(t_k,x,u) \to y \right\}.$$

For the following proposition see [16, Proposition 4.2.7, p. 98].

1.3.15 Proposition:
Consider the control-affine system (1.34) *and let* $\mathcal{U}$ *be endowed with the weak***-topology. Then the shift flow $\Theta : \mathbb{R} \times \mathcal{U} \to \mathcal{U}$ is topologically mixing, topologically transitive and chain transitive.*

[14] The weak*-topology on $L^\infty(\mathbb{R},\mathbb{R}^m)$ is the weakest topology such that for all $v \in L^1(\mathbb{R},\mathbb{R}^m)$ the functional $u \mapsto \int_{\mathbb{R}} \langle v(t), u(t) \rangle dt$, $L^\infty(\mathbb{R},\mathbb{R}^m) \to \mathbb{R}$, is continuous.

Chapter 2

Basics of Invariance Entropy

In this chapter, we introduce the central notion of this thesis—invariance entropy. Roughly speaking, invariance entropy is a quantity that measures how often open loop controls have to be updated in order to keep a continuous-time control system in a given compact and controlled invariant subset Q of the state space, starting from a set $K \subset Q$. The notion, originally introduced in Colonius & Kawan [15], is defined in a fashion similar to the Bowen-Dinaburg characterization of topological entropy for dynamical systems via (n, ε)-spanning sets (see Bowen [9]). We consider for each positive time $T > 0$ the minimal number $r^*_{\mathrm{inv}}(T, K, Q)$ of control functions necessary to keep every trajectory starting in K in the bigger set Q up to time T. Then we consider the exponential growth rate of these numbers as T tends to infinity, i.e., we define

$$h^*_{\mathrm{inv}}(K, Q) := \limsup_{T\to\infty} \frac{1}{T} \ln r^*_{\mathrm{inv}}(T, K, Q).$$

This number, called the *strict invariance entropy* of (K, Q), has the disadvantage that it may be infinite. In fact, the numbers $r^*_{\mathrm{inv}}(T, K, Q)$ may not be finite. To overcome this problem, we define a weaker version, simply called *invariance entropy*, in the following way: For $T, \varepsilon > 0$ the minimal number of control functions necessary to keep every trajectory with initial value in K in an ε-neighborhood of Q is denoted by $r_{\mathrm{inv}}(T, \varepsilon, K, Q)$. Then again, the exponential growth rate for $T \to \infty$ is considered and in the end, ε is sent to zero. Precisely,

$$h_{\mathrm{inv}}(K, Q) := \lim_{\varepsilon\searrow 0} \limsup_{T\to\infty} \frac{1}{T} \ln r_{\mathrm{inv}}(T, \varepsilon, K, Q).$$

For the numbers $r_{\mathrm{inv}}(T, \varepsilon, K, Q)$ finiteness can easily be shown. Indeed, as we will see in Chapter 3, also $h_{\mathrm{inv}}(K, Q)$ is finite. $h_{\mathrm{inv}}(K, Q)$ is less than or equal to $h^*_{\mathrm{inv}}(K, Q)$, but it is not clear if equality holds in case $h^*_{\mathrm{inv}}(K, Q)$ is finite. In Chapter 4, we will prove equality in two special situations, but in general we have to leave this problem open.

In this chapter, we also show that invariance entropy shares several properties with topological entropy. In particular, invariance entropy is preserved under a continuous change of coordinates in the state space.[1] Then we extend the concept of invariance entropy to certain noncompact controlled invariant subsets of the state space, which can be projected to compact subsets in the state space of another control system, which is semiconjugate to the given one via the projection map. As an example, we introduce bilinear control systems on $\mathbb{R}^d$ and their projections to the unit sphere S^{d-1}.

Finally, we ask the question what we can say about the invariance entropy of a control-affine system under certain assumptions on the behavior of the system in a neighborhood of the set Q. Here we find that under the assumption that Q is isolated (in the sense that every solution which stays in a given neighborhood of Q for all times must already be contained in Q), the invariance entropy is given by $h_{\mathrm{inv}}(\varepsilon, K, Q)$ for small ε and hence taking the limit $\varepsilon \searrow 0$ is unnecessary. If we instead assume that controllability holds in a neighborhood of Q (in a sense which has to be made precise), then it turns out that the limes superior in the definition of $h_{\mathrm{inv}}(K, Q)$ can be replaced by a limes inferior. Similar results are known for the topological entropy of a dynamical system (see Section 2.3 for more details).

2.1 Definition

Now we will define the notions of invariance entropy and strict invariance entropy and derive some basic properties of these quantities. Throughout this section, we consider control system (1.7) and assume that $K, Q \subset M$ are nonvoid compact sets with $K \subset Q$ and Q being controlled invariant.

2.1.1 Definition (Strict Invariance Entropy):
*For given $T > 0$ a set $\mathcal{S}^* \subset \mathcal{U}$ is called **T-spanning** for (K, Q) if*

$$\forall x \in K : \ \exists u \in \mathcal{S}^* : \ \forall t \in [0, T] : \ \varphi(t, x, u) \in Q.$$

*By $r^*_{\mathrm{inv}}(T, K, Q)$ we denote the number of elements in a minimal T-spanning set for (K, Q). If there exists no finite T-spanning set, we set $r^*_{\mathrm{inv}}(T, K, Q) := \infty$. The **strict invariance entropy** of (K, Q) is defined by*

$$h^*_{\mathrm{inv}}(K, Q) := \limsup_{T \to \infty} \frac{1}{T} \ln r^*_{\mathrm{inv}}(T, K, Q) \tag{2.1}$$

if $r^*_{\mathrm{inv}}(T, K, Q)$ is finite for all $T > 0$. Otherwise $h^*_{\mathrm{inv}}(K, Q) := \infty$.

Hence, the strict invariance entropy is defined as the exponential growth rate of the minimal number of control functions necessary to keep every trajectory starting in K in the bigger set Q up to time T, as T tends to infinity. One problem with this definition is that one may need more than finitely many control functions in order to keep all trajectories in Q for some positive time, i.e., the numbers $r^*_{\mathrm{inv}}(T, K, Q)$ may not be finite. Indeed, the following examples show that this can happen.

[1] In the context of dynamical systems this is known under the name “topological conjugacy”. In the context of control systems it is usually called “state equivalence”.

2.1.2 Example:

Consider the one-dimensional linear control system

$$\dot{x}(t) = -x(t) + u(t), \quad u \in \mathcal{U},$$

with control range $U = [-1,1]$. Let $K = Q \subset [-1,1]$ be an infinite compact set which is totally disconnected, e.g., a Cantor set. Then for every $x \in Q$ there exists a unique constant control function $u_x \in \mathcal{U}$ with $\varphi(t,x,u_x) = x$ for all $t \geq 0$, namely $u_x(t) \equiv x$. Hence, Q is controlled invariant. Since Q is totally disconnected, each point $x \in Q$ can be kept in Q up to some positive time $T > 0$ only by choosing the control function u_x, which makes x an equilibrium. Consequently, since Q has infinitely many elements, no finite T-spanning set for (Q,Q) exists. Hence, we have $r^*_{\text{inv}}(T,Q,Q) = \infty$ for all $T > 0$ and therefore $h^*_{\text{inv}}(Q,Q) = \infty$. ◊

2.1.3 Example:

Consider the control system

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \left(\frac{x(t)}{(x(t)^2 + y(t)^2)^{1/2}} - u(t)\right)^2 \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad u \in \mathcal{U},$$

on $M := \mathbb{R}^2 \backslash \{(0,0)\}$ with control range $U = [-1,1]$. For every $z = (x,y) \in M$ there exists a constant control function $u_z \in \mathcal{U}$ such that z becomes an equilibrium, namely $u_z(t) \equiv \frac{x}{(x^2+y^2)^{1/2}}$. Hence, every subset of M is controlled invariant. We define

$$Q := \left\{(x,y) \in M \mid \tfrac{1}{2} \leq \|(x,y)\| \leq 1\right\},$$

i.e., Q is the compact annulus with inner radius $\frac{1}{2}$ and outer radius 1. Obviously, every point $z \in M$ can only be steered along the line through $(0,0)$ and z away from the origin. Hence, a point z on the outer boundary $S_1(0) \subset Q$ can only be kept in Q for some positive time $T > 0$ with the constant control function u_z. Since one needs infinitely many of these control functions for all points on $S_1(0)$, there exists no finite T-spanning set for $(S_1(0), Q)$ and hence for Q. This proves that $h^*_{\text{inv}}(Q,Q) = \infty$. ◊

2.1.4 Remark:

The system in the preceding example is obviously not locally accessible but this is not what causes the problem. The problem also appears for the system

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \left(\frac{x(t)}{(x(t)^2 + y(t)^2)^{1/2}} - u(t)\right)^2 \begin{pmatrix} \cos v(t) & -\sin v(t) \\ \sin v(t) & \cos v(t) \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

with $(u(t), v(t)) \in [-1,1] \times [-\frac{\pi}{4}, \frac{\pi}{4}]$, because it also has the property that every point on the outer boundary of the annulus Q can only be steered in directions which point to the outside of Q. Local accessibility of this system easily follows from Theorem 1.3.5 (Krener's Theorem), since the right-hand side is of class C^∞ and the admissible directions from each point span the whole tangent space.

In order to avoid the problem of dealing with infinities, we introduce a second, weaker version of invariance entropy. In order to define this quantity, we use an arbitrary but fixed metric d on M, which induces the given topology:[2]

2.1.5 Definition (Invariance Entropy):
For given $T, \varepsilon > 0$ we call a set $\mathcal{S} \subset \mathcal{U}$ **($\boldsymbol{T}, \boldsymbol{\varepsilon}$)-spanning** *for (K, Q) if*

$$\forall x \in K : \exists u \in \mathcal{S} : \forall t \in [0, T] : \exists y \in Q : d(\varphi(t, x, u), y) < \varepsilon.$$

By $r_{\text{inv}}(T, \varepsilon, K, Q)$ we denote the number of elements in a minimal (T, ε)-spanning set for (K, Q). For every $\varepsilon > 0$ we define

$$h_{\text{inv}}(\varepsilon, K, Q) := \limsup_{T\to\infty} \frac{1}{T} \ln r_{\text{inv}}(T, \varepsilon, K, Q).$$

The **invariance entropy** *of (K, Q) is then given by*

$$h_{\text{inv}}(K, Q) := \lim_{\varepsilon \searrow 0} h_{\text{inv}}(\varepsilon, K, Q). \tag{2.2}$$

2.1.6 Remarks:

- Note that the notion of (T, ε)-spanning sets differs from the one used for defining topological entropy (see Section A.2).
- One could do without the metric by replacing the ε-neighborhoods of Q by arbitrary open neighborhoods and taking the supremum over all neighborhoods instead of letting ε tend to 0 (see also Proposition 2.1.10(i)).
- Instead of the natural logarithm it would actually be more appropriate to take the logarithm to the base 2, since we want to measure information. But as we will see, the formulas for the invariance entropy become nicer when working with the natural logarithm. (Otherwise the number $\log_2 e$ would explicitly appear as a factor in every formula.)
- In the case $K = Q$ we often suppress the argument K in $r^*_{\text{inv}}(\cdot)$, $r_{\text{inv}}(\cdot)$, $h_{\text{inv}}(\cdot)$ and $h^*_{\text{inv}}(\cdot)$. That means, we write, e.g., $h_{\text{inv}}(Q)$ instead of $h_{\text{inv}}(Q, Q)$. Sometimes we stress the (possible) dependence of $h_{\text{inv}}(K, Q)$ on the control range U or on the metric d or on the right-hand side F by writing $h_{\text{inv}}(K, Q; U)$, $h_{\text{inv}}(K, Q; d)$ or $h_{\text{inv}}(K, Q; F)$.
- Note that for the definition of (strict) invariance entropy we do not need to assume that all maximal solutions of the control system are defined on $\mathbb{R}$, since we only have to consider solutions which do not leave a small neighborhood of a compact set, which, if chosen small enough, has a compact closure (see Lemma A.3.2).
- In the definition of invariance entropy the requirement of the control range to be compact is not essential. Indeed, many results of this and the following chapters also hold true for systems with noncompact control range.

The following propositions summarize the most elementary properties of the quantities $r_{\text{inv}}(\cdot)$, $r^*_{\text{inv}}(\cdot)$, $h_{\text{inv}}(\cdot)$ and $h^*_{\text{inv}}(\cdot)$.

[2] Note that d need not be induced by a Riemannian metric.

2.1.7 Proposition:
The number $r_{\text{inv}}(T,\varepsilon,K,Q)$ is finite for all $T,\varepsilon>0$.

Proof:
Let $T,\varepsilon>0$ be given. Since Q is controlled invariant and $K\subset Q$, for every $x\in K$ there exists a control function $u_x\in\mathcal{U}$ such that $\varphi([0,T],x,u_x)\subset Q$. Since $N_\varepsilon(Q)$ is open in M, by continuous dependence on the initial value (Corollary 1.2.11) there exists a neighborhood W_x of x such that $\varphi([0,T],W_x,u_x)\subset N_\varepsilon(Q)$. By compactness, K can be covered by finitely many of these neighborhoods, i.e., there exist points $x_1,\dots,x_n\in K$ ($n\in\mathbb{N}$) with $K\subset\bigcup_{i=1}^n W_{x_i}$. Hence, the set $\mathcal{S}:=\{u_{x_1},\dots,u_{x_n}\}$ is a finite (T,ε)-spanning set for (K,Q). □

The next proposition summarizes monotonicity properties. The proof immediately follows from the definitions and hence we omit it.

2.1.8 Proposition:

(i) *If $T_1<T_2$, then $r^*_{\text{inv}}(T_1,K,Q)\le r^*_{\text{inv}}(T_2,K,Q)$ and $r_{\text{inv}}(T_1,\varepsilon,K,Q)\le r_{\text{inv}}(T_2,\varepsilon,K,Q)$ for every $\varepsilon>0$.*

(ii) *If $\varepsilon_1<\varepsilon_2$, then $r_{\text{inv}}(T,\varepsilon_1,K,Q)\ge r_{\text{inv}}(T,\varepsilon_2,K,Q)$ for all $T>0$ and hence $h_{\text{inv}}(\varepsilon_1,K,Q)\ge h_{\text{inv}}(\varepsilon_2,K,Q)$.*

(iii) *If $K_1\subset K_2$, then $h_{\text{inv}}(K_1,Q)\le h_{\text{inv}}(K_2,Q)$ and $h^*_{\text{inv}}(K_1,Q)\le h^*_{\text{inv}}(K_2,Q)$.*

(iv) *Assume that $Q_1\subset Q_2$ are controlled invariant and $K\subset Q_1$. Then $h^*_{\text{inv}}(K,Q_1)\ge h^*_{\text{inv}}(K,Q_2)$ and $h_{\text{inv}}(K,Q_1)\ge h_{\text{inv}}(K,Q_2)$.*

(v) *Consider two different (compact) control ranges $U_1,U_2\subset\mathbb{R}^m$ with $U_1\subset U_2$. Assume that Q is controlled invariant with respect to both U_1 and U_2. Then $h^*_{\text{inv}}(K,Q;U_1)\ge h^*_{\text{inv}}(K,Q;U_2)$ and $h_{\text{inv}}(K,Q;U_1)\ge h_{\text{inv}}(K,Q;U_2)$.*

From Proposition 2.1.7 and 2.1.8(ii) it follows that the limit in (2.2) exists and thus the following is true.

2.1.9 Corollary:
$h_{\text{inv}}(K,Q)$ is a well-defined number contained in the interval $[0,\infty]$.

2.1.10 Proposition:

(i) *$h_{\text{inv}}(K,Q)$ does not depend on the metric d.*

(ii) *The number $r^*_{\text{inv}}(T,Q)$ is either finite for all $T>0$ or for none.*

(iii) *The function $T\mapsto\ln r^*_{\text{inv}}(T,Q)$, $(0,\infty)\to\mathbb{R}^+_0$, is subadditive and therefore*

$$\boxed{h^*_{\text{inv}}(Q)=\lim_{T\to\infty}\frac{1}{T}\ln r^*_{\text{inv}}(T,Q)=\inf_{T>0}\frac{1}{T}\ln r^*_{\text{inv}}(T,Q).}\tag{2.3}$$

(iv) *$r_{\text{inv}}(T,\varepsilon,K,Q)\le r^*_{\text{inv}}(T,K,Q)$ for all $T,\varepsilon>0$ and hence*

$$\boxed{h_{\text{inv}}(K,Q)\le h^*_{\text{inv}}(K,Q).}\tag{2.4}$$

Proof:

(i) Let $\tilde{d}$ be another metric on M, which induces the given topology. Since Q is compact, the identity map $\mathrm{id} : (M,d) \to (M,\tilde{d})$ is uniformly continuous on Q, i.e.,

$$\forall \varepsilon > 0 : \ \exists \delta > 0 : \ \forall x \in Q : \ \forall y \in M : \ d(x,y) < \delta \Rightarrow \tilde{d}(x,y) < \varepsilon.$$

Consequently, the δ-neighborhood of Q with respect to d is contained in the ε-neighborhood with respect to $\tilde{d}$. This implies

$$r_{\mathrm{inv}}(T,\varepsilon,K,Q;\tilde{d}) \leq r_{\mathrm{inv}}(T,\delta,K,Q;d) \quad \text{for all } T > 0$$

and hence $h_{\mathrm{inv}}(\varepsilon,K,Q;\tilde{d}) \leq h_{\mathrm{inv}}(\delta,K,Q;d)$. Since

$$h_{\mathrm{inv}}(\delta,K,Q;d) \leq h_{\mathrm{inv}}(K,Q;d) \quad \text{for all } \delta > 0,$$

we obtain $h_{\mathrm{inv}}(K,Q;\tilde{d}) \leq h_{\mathrm{inv}}(K,Q;d)$. Changing the roles of d and $\tilde{d}$ yields the result.

(ii) Assume that $r^*_{\mathrm{inv}}(T_0,Q) < \infty$ for some $T_0 > 0$. By Proposition 2.1.8(i) we have $r^*_{\mathrm{inv}}(T,Q) \leq r^*_{\mathrm{inv}}(T_0,Q) < \infty$ for all $T \in (0,T_0)$. Now let $T > T_0$. Choose $k \in \mathbb{N}$ such that $kT_0 \geq T$ and let $\mathcal{S}^* = \{u_1,\ldots,u_n\}$ be a minimal T_0-spanning set for Q. For every k-tuple $(i_0,\ldots,i_{k-1}) \in \{1,\ldots,n\}^k$ we define a control function $u_{i_0,i_1,\ldots,i_{k-1}} \in \mathcal{U}$ by

$$u_{i_0,i_1,\ldots,i_{k-1}}(t) := u_{i_j}(t - jT_0) \text{ for all } t \in [jT_0,(j+1)T_0), \ j = 0,1,\ldots,k-1.$$

On $\mathbb{R}\backslash[0,kT_0)$ the function may be extended arbitrarily. Altogether we obtain n^k control functions by this construction. Now consider the set

$$\mathcal{S}^*_k := \left\{ u_{i_0,i_1,\ldots,i_{k-1}} \ : \ (i_0,i_1,\ldots,i_{k-1}) \in \{1,\ldots,n\}^k \right\}.$$

Let $x_0 \in Q$ be an arbitrary point. Since $\mathcal{S}^*$ is strictly T_0-spanning, there exists $u_{i_0} \in \mathcal{S}^*$ with $\varphi([0,T_0],x_0,u_{i_0}) \subset Q$. Let $x_1 := \varphi(T_0,x_0,u_{i_0})$. Then there exists $u_{i_1} \in \mathcal{S}^*$ with $\varphi([0,T_0],x_1,u_{i_1}) \subset Q$. Again, for $x_2 := \varphi(T_0,x_1,u_{i_1})$ there exists $u_{i_2} \in \mathcal{S}^*$ and so on. After k steps we obtain (by the cocycle property (1.13)) $u_{i_0},u_{i_1},\ldots,u_{i_{k-1}} \in \mathcal{S}^*$ with

$$\varphi([0,kT_0],x_0,u_{i_0,i_1,\ldots,i_{k-1}}) \subset Q.$$

This implies that $\mathcal{S}^*_k$ is a (kT_0)-spanning set for Q and thus

$$r^*_{\mathrm{inv}}(T,Q) \leq r^*_{\mathrm{inv}}(kT_0,Q) \leq \#\mathcal{S}^*_k = n^k < \infty,$$

which proves the assertion.

(iii) If $r^*_{\mathrm{inv}}(T,Q) = \infty$ for all $T > 0$, the assertion is trivial. So assume $r^*_{\mathrm{inv}}(T,Q) < \infty$ for all $T > 0$. If we can show that $T \mapsto \ln r^*_{\mathrm{inv}}(T,Q)$ is subadditive, then (2.3) follows from Lemma A.3.7. In order to show subadditivity, let $T_1, T_2 > 0$ be given. Let $\mathcal{S}_j$ be a minimal T_j-spanning set for Q, $j = 1, 2$. Define control functions

$$u_{vw}(t) := \begin{cases} v(t) & \text{for } t \in [0, T_1] \\ w(t - T_1) & \text{for } t \in (T_1, T_1 + T_2] \end{cases}, \quad (v, w) \in \mathcal{S}_1 \times \mathcal{S}_2.$$

Extend u_{vw} arbitrarily on $\mathbb{R} \backslash [0, T_1 + T_2]$. Then $\mathcal{S}_{12} := \{u_{vw} \mid (v, w) \in \mathcal{S}_1 \times \mathcal{S}_2\}$ is a $(T_1 + T_2)$-spanning set for Q, which follows with the same arguments as in the proof of (ii). Hence,

$$r^*_{\mathrm{inv}}(T_1 + T_2, Q) \leq \#\mathcal{S}_{12} = \#\mathcal{S}_1 \cdot \#\mathcal{S}_2 = r^*_{\mathrm{inv}}(T_1, Q) \cdot r^*_{\mathrm{inv}}(T_2, Q).$$

Applying the logarithm to this inequality yields the result.

(iv) Since $Q \subset N_\varepsilon(Q)$ for all $\varepsilon > 0$, every T-spanning set is also (T, ε)-spanning for all $\varepsilon > 0$. This implies the assertion.

□

The following corollary is an immediate consequence of Proposition 2.1.10(ii) and (iii).

2.1.11 Corollary:
The following statements are equivalent:

(a) $r^*_{\mathrm{inv}}(T, Q)$ *is finite for some* $T > 0$.

(b) $r^*_{\mathrm{inv}}(T, Q)$ *is finite for all* $T > 0$.

(c) $h^*_{\mathrm{inv}}(Q)$ *is finite.*

2.1.12 Remark:
In Section 3.1, we will show that, in contrast to the strict invariance entropy, the invariance entropy $h_{\mathrm{inv}}(K, Q)$ is always finite. The next proposition thus yields two conditions for finiteness of $h^*_{\mathrm{inv}}(K, Q)$.

2.1.13 Proposition:

(i) *Assume that M is diffeomorphic to $\mathbb{R}$ or S^1 and Q is a compact interval. Then $h^*_{\mathrm{inv}}(K, Q) < \infty$.*

(ii) *Assume that also K is controlled invariant and Q is a neighborhood of K. Then $h^*_{\mathrm{inv}}(K, Q) \leq h_{\mathrm{inv}}(K)$.*

Proof:

(i) It suffices to prove the assertion for $K = Q$, since $h^*_{\mathrm{inv}}(K, Q) \leq h^*_{\mathrm{inv}}(Q)$ by Proposition 2.1.8(iii). Without loss of generality we may assume that $M = \mathbb{R}$. Let $Q = [a, b]$. By Corollary 2.1.11 it suffices to show that $r^*_{\mathrm{inv}}(T, Q) < \infty$ for some $T > 0$. If $a = b$, then there must exist a constant control function which makes a an equilibrium point. This implies $r^*_{\mathrm{inv}}(T, Q) = 1$ for all $T > 0$. Otherwise there exists $c \in (a, b)$. Since Q is controlled invariant, there are $u_a, u_b \in \mathcal{U}$ with $\varphi(\mathbb{R}^+_0, a, u_a) \subset Q$ and $\varphi(\mathbb{R}^+_0, b, u_b) \subset Q$. Since (a, b) is open and solutions depend continuously on the initial value, there exist times $T_a, T_b > 0$ with $\varphi([0,T_a],[a,c],u_a) \subset Q$ and $\varphi([0,T_b],[c,b],u_b) \subset Q$. Hence, for $T := \min\{T_a, T_b\}$ the set $\mathcal{S}^* := \{u_a, u_b\}$ is T-spanning for Q.

(ii) Since Q is a neighborhood of K, there exists $\varepsilon_0 > 0$ with $N_{\varepsilon_0}(K) \subset Q$. Let $\mathcal{S}$ be a minimal (T,ε_0)-spanning set for (K,K). Then obviously $\mathcal{S}$ is also T-spanning for (K,Q) which implies $r^*_{\text{inv}}(T,K,Q) \leq r_{\text{inv}}(T,\varepsilon_0,K)$. Since this is true for all $T > 0$, we obtain $h^*_{\text{inv}}(K,Q) \leq h_{\text{inv}}(\varepsilon,K)$ for all $\varepsilon \in (0,\varepsilon_0)$. For ε tending to zero $h^*_{\text{inv}}(K,Q) \leq h_{\text{inv}}(K)$ follows.

□

The following proposition yields a sufficient condition for the equality of $h_{\text{inv}}(K,Q)$ and $h^*_{\text{inv}}(K,Q)$ in terms of the numbers $r_{\text{inv}}(T,\varepsilon,K,Q)$. It uses the fact that for control-affine systems one has $r_{\text{inv}}(T,\varepsilon,K,Q) \to r^*_{\text{inv}}(T,K,Q)$ for $\varepsilon \searrow 0$. The condition formulated in the proposition implies uniform convergence (in T) of $\frac{1}{T}\ln r_{\text{inv}}(T,\varepsilon,K,Q)$ to $\frac{1}{T} r^*_{\text{inv}}(T,K,Q)$ for $\varepsilon \searrow 0$, which is sufficient for the two limits in the definition of $h_{\text{inv}}(K,Q)$ to be commutable.

2.1.14 Proposition:
Consider the control-affine system (1.34). *Let $Q \subset M$ be a compact controlled invariant set and $K \subset Q$ compact. Moreover, assume that*

$$\begin{aligned}\forall \delta > 0 : \ \exists \varepsilon_0 > 0 : \forall \varepsilon_1 \in (0,\varepsilon_0] : \ \exists T_0 > 0 : \ \forall \varepsilon_2 \in (0,\varepsilon_1) : \ \forall T \geq T_0 : \\ r_{\text{inv}}(T,\varepsilon_2,K,Q) \leq e^{\delta T} r_{\text{inv}}(T,\varepsilon_1,K,Q).\end{aligned} \tag{2.5}$$

If, in addition, $h^*_{\text{inv}}(Q) < \infty$, then $h_{\text{inv}}(K,Q) = h^*_{\text{inv}}(K,Q)$.

Proof:
We subdivide the proof into two parts.

Step 1: We show that for each fixed $T > 0$ it holds that

$$\lim_{\varepsilon \searrow 0} r_{\text{inv}}(T,\varepsilon,K,Q) = r^*_{\text{inv}}(T,K,Q).$$

By the assumption $h^*_{\text{inv}}(Q) < \infty$ and Corollary 2.1.11 we know that $r^*_{\text{inv}}(T,K,Q) \leq r^*_{\text{inv}}(T,Q) < \infty$. Since $r_{\text{inv}}(T,\varepsilon,K,Q) \leq r^*_{\text{inv}}(T,K,Q)$ for all $\varepsilon > 0$ and $r_{\text{inv}}(T,\varepsilon_1,K,Q) \geq r_{\text{inv}}(T,\varepsilon_2,K,Q)$ for $\varepsilon_1 < \varepsilon_2$, the function

$$\varepsilon \mapsto r_{\text{inv}}(T,\varepsilon,K,Q), \quad (0,\infty) \to \mathbb{N},$$

is monotonically decreasing and bounded from above by $r^*_{\text{inv}}(T,K,Q)$. This implies that the limit $\lim_{\varepsilon \searrow 0} r_{\text{inv}}(T,\varepsilon,K,Q)$ exists and is not greater than $r^*_{\text{inv}}(T,K,Q)$. Hence, there are $n_0, r \in \mathbb{N}$ such that

$$r_{\text{inv}}(T,\tfrac{1}{n},K,Q) = r \leq r^*_{\text{inv}}(T,K,Q) \quad \text{for all } n \geq n_0.$$

For each integer $n \geq n_0$ let $\mathcal{S}_n$ denote a minimal $(T,\frac{1}{n})$-spanning set for (K,Q),

$$\mathcal{S}_n = \{v_1(n), \dots, v_r(n)\}.$$

By Proposition 1.3.14 $\mathcal{U}$ is compact in the weak*-topology. Hence, we can choose converging subsequences $v_j(m_n^{(j)}) \to v_j^*$ for $j = 1,\dots,r$. By a standard construction[3] we can assume without loss of generality that the subsequences $(m_n^{(j)})_{n\in\mathbb{N}}$ are identical for $j=1,\dots,r$, say $m_n^{(j)} \equiv m_n$. Set

$$\mathcal{S}^* := \{v_1^*, \dots, v_r^*\}.$$

We want to show that $\mathcal{S}^*$ is a T-spanning set for (K,Q). To this end, pick $x \in K$ arbitrarily. Then for every integer $n \in \mathbb{N}$ with $m_n \geq n_0$ there exists $j = j(n) \in \{1,\dots,r\}$ such that

$$\varphi([0,T], x, v_{j(n)}(m_n)) \subset N_{\frac{1}{m_n}}(Q). \tag{2.6}$$

Since $j(n)$ can only vary within the finite set $\{1,\dots,r\}$, there must be one $j_0 \in \{1,\dots,r\}$ such that $j(n) = j_0$ for infinitely many $n \in \mathbb{N}$. Let $(m_{k_n})_{n\in\mathbb{N}}$ be a corresponding subsequence. By continuity of φ (Proposition 1.3.14) this implies for every $t \in [0,T]$ that

$$\begin{aligned}\lim_{n\to\infty} \varphi(t,x,v_{j(k_n)}(m_{k_n})) &= \lim_{n\to\infty} \varphi(t,x,v_{j_0}(m_{k_n})) = \varphi(t,x,\lim_{n\to\infty} v_{j_0}(m_{k_n}))\\ &= \varphi(t,x,\lim_{n\to\infty} v_{j_0}(m_n)) = \varphi(t,x,v_{j_0}^*).\end{aligned}$$

By (2.6) and compactness of Q it follows that $\varphi(t,x,v_{j_0}^*) \in Q$. This proves that $\mathcal{S}^*$ is T-spanning for (K,Q) and hence

$$r_{\mathrm{inv}}^*(T,K,Q) \leq \#\mathcal{S}^* = r = \lim_{\varepsilon\searrow 0} r_{\mathrm{inv}}(T,\varepsilon,K,Q) \leq r_{\mathrm{inv}}^*(T,K,Q).$$

<u>*Step 2:*</u> Assumption (2.5) can also be written as

$$\begin{aligned}&\forall \delta > 0 : \ \exists \varepsilon_0 > 0 : \ \forall \varepsilon_1 \in (0,\varepsilon_0] : \ \exists T_0 > 0 : \ \forall \varepsilon_2 \in (0,\varepsilon_1) : \ \forall T \geq T_0 : \\ &\qquad \left|\tfrac{1}{T}\ln r_{\mathrm{inv}}(T,\varepsilon_2,K,Q) - \tfrac{1}{T}\ln r_{\mathrm{inv}}(T,\varepsilon_1,K,Q)\right| \leq \delta.\end{aligned}$$

For arbitrary $\delta > 0$ take $\varepsilon_0 = \varepsilon_0(\frac{\delta}{2})$ as above, and let $\varepsilon \in (0,\varepsilon_0]$ be arbitrary. Then take $T_0 = T_0(\frac{\delta}{2},\varepsilon_0,\varepsilon)$ as above. For every $T \geq T_0$ Step 1 yields $\tilde{\varepsilon}(T) \in (0,\varepsilon)$ with

$$\left|\tfrac{1}{T}\ln r_{\mathrm{inv}}^*(T,K,Q) - \tfrac{1}{T}\ln r_{\mathrm{inv}}(T,\tilde{\varepsilon}(T),K,Q)\right| \leq \tfrac{\delta}{2}. \tag{2.7}$$

Using the triangle inequality we obtain for all $T \geq T_0$:

$$\begin{aligned}&\left|\tfrac{1}{T}\ln r_{\mathrm{inv}}^*(T,K,Q) - \tfrac{1}{T}\ln r_{\mathrm{inv}}(T,\varepsilon,K,Q)\right|\\ \leq\ &\left|\tfrac{1}{T}\ln r_{\mathrm{inv}}^*(T,K,Q) - \tfrac{1}{T}\ln r_{\mathrm{inv}}(T,\tilde{\varepsilon}(T),K,Q)\right|\\ +\ &\left|\tfrac{1}{T}\ln r_{\mathrm{inv}}(T,\tilde{\varepsilon}(T),K,Q) - \tfrac{1}{T}\ln r_{\mathrm{inv}}(T,\varepsilon,K,Q)\right| \leq \tfrac{\delta}{2} + \tfrac{\delta}{2} = \delta.\end{aligned}$$

Hence, we have proved the following:

$$\begin{aligned}&\forall \delta > 0 : \ \exists \varepsilon_0 > 0 : \ \forall \varepsilon \in (0,\varepsilon_0] : \ \exists T_0 > 0 : \ \forall T \geq T_0 : \\ &\qquad \left|\tfrac{1}{T}\ln r_{\mathrm{inv}}^*(T,K,Q) - \tfrac{1}{T}\ln r_{\mathrm{inv}}(T,\varepsilon,K,Q)\right| \leq \delta.\end{aligned}$$

[3] First take a converging subsequence $v_1(m_n)$ of $v_1(n)$. Then take a converging subsequence of $v_2(m_n)$ and so on.

This implies

$$\begin{aligned}&\forall \delta > 0: \quad \exists \varepsilon_0 > 0: \quad \forall \varepsilon \in (0, \varepsilon_0]: \\ &\qquad \limsup_{T\to\infty} \left| \tfrac{1}{T} \ln r^*_{\mathrm{inv}}(T,K,Q) - \tfrac{1}{T} \ln r_{\mathrm{inv}}(T,\varepsilon,K,Q) \right| \leq \delta.\end{aligned}$$

Now for $\delta > 0$ take $\varepsilon_0 = \varepsilon_0(\delta)$ as above and let $\varepsilon \in (0, \varepsilon_0]$. For brevity we write $f(T) := \frac{1}{T} \ln r^*_{\mathrm{inv}}(T,K,Q)$ and $g(T) := \frac{1}{T} \ln r_{\mathrm{inv}}(T,\varepsilon,K,Q)$. Then we have

$$\begin{aligned}|h^*_{\mathrm{inv}}(K,Q) - h_{\mathrm{inv}}(\varepsilon,K,Q)| &= \left| \limsup_{T\to\infty} f(T) - \limsup_{T\to\infty} g(T) \right| \\ &= \left| \lim_{T_0\to\infty} \sup_{T\geq T_0} f(T) - \lim_{T_0\to\infty} \sup_{T\geq T_0} g(T) \right| \\ &= \left| \lim_{T_0\to\infty} \left[\sup_{T\geq T_0} f(T) - \sup_{T\geq T_0} g(T) \right] \right| \\ &= \lim_{T_0\to\infty} \left[\sup_{T\geq T_0} f(T) - \sup_{T\geq T_0} g(T) \right].\end{aligned}$$

The last equality follows from the fact that $f(T) \geq g(T)$ for all $T > 0$. Now we use that

$$\sup_{T\geq T_0} f(T) - \sup_{T\geq T_0} g(T) \leq \sup_{T\geq T_0} \left[f(T) - g(T) \right],$$

which follows from

$$\sup_{T\geq T_0} f(T) = \sup_{T\geq T_0} \left[(f(T) - g(T)) + g(T) \right] \leq \sup_{T\geq T_0} \left[f(T) - g(T) \right] + \sup_{T\geq T_0} g(T).$$

Hence, we obtain

$$\begin{aligned}|h^*_{\mathrm{inv}}(K,Q) - h_{\mathrm{inv}}(\varepsilon,K,Q)| &\leq \lim_{T_0\to\infty} \sup_{T\geq T_0} \left[f(T) - g(T) \right] \\ &= \limsup_{T\to\infty} \left[f(T) - g(T) \right] \\ &= \limsup_{T\to\infty} |f(T) - g(T)| \leq \delta.\end{aligned}$$

This proves the assertion. □

2.1.15 Open Question:
Does $h^*_{\mathrm{inv}}(K,Q) < \infty$ imply $h_{\mathrm{inv}}(K,Q) = h^*_{\mathrm{inv}}(K,Q)$?

2.2 Elementary Properties

In this section, we prove more elementary properties of the (strict) invariance entropy. The first proposition yields a sufficient condition for the strict invariance entropy to vanish.

2.2.1 Proposition:
Consider the control system (1.7). *Let $Q \subset M$ be a compact controlled invariant set, and $K \subset Q$ compact. If there exist finitely many control functions $u_1, \ldots, u_n \in \mathcal{U}$ such that for all $x \in K$ there is $i \in \{1, \ldots, n\}$ with $\varphi(\mathbb{R}_0^+, x, u_i) \subset Q$, then $h^*_{\text{inv}}(K, Q) = 0$. In particular, this condition is satisfied in each of the following situations:*

(i) *K is finite.*

(ii) *Q is positively invariant (in particular, $Q = M$).*

(iii) *There exists some $u_0 \in U$ with $F(x, u_0) = 0$ for all $x \in K$.*

Proof:
Under the assumption it is clear that $r^*_{\text{inv}}(T, K, Q) \leq n$ for all $T > 0$, which implies

$$h^*_{\text{inv}}(K, Q) \leq \limsup_{T \to \infty} \frac{\ln n}{T} = 0.$$

If K is finite, then this condition immediately follows from the controlled invariance of Q. If Q is positively invariant, then $\varphi(\mathbb{R}_0^+, K, u) \subset Q$ for all $u \in \mathcal{U}$. In case (iii) every point $x \in K$ becomes an equilibrium if we choose the constant control function $u(t) \equiv u_0$ and hence the condition is satisfied with $n = 1$.
□

The resemblance of the definition of invariance entropy to the Bowen-Dinaburg characterization of topological entropy via (n, ε)-spanning sets suggests that both quantities have similar properties. Indeed, the following propositions affirm this conjecture.

2.2.2 Proposition:
Consider control system (1.7). *Let $Q \subset M$ be a compact controlled invariant set, and $K \subset Q$ compact. Consider for some $s > 0$ also the control system*

$$\dot{x}(t) = s \cdot F(x(t), u(t)), \quad u \in \mathcal{U}. \tag{2.8}$$

Then Q is also controlled invariant with respect to (2.8). *For the corresponding invariance entropies the following holds for all $\varepsilon > 0$:*

$$\boxed{h_{\text{inv}}(\varepsilon, K, Q; sF) = s \cdot h_{\text{inv}}(\varepsilon, K, Q; F), \quad h^{(*)}_{\text{inv}}(K, Q; sF) = s \cdot h^{(*)}_{\text{inv}}(K, Q; F).}$$

Proof:
Let φ_s denote the cocycle of system (2.8). Let $(x, u) \in M \times \mathcal{U}$ and define $\tilde{u} : \mathbb{R} \to \mathbb{R}^m$ by $\tilde{u}(t) := u(ts)$ for all $t \in \mathbb{R}$. Then obviously $\tilde{u} \in \mathcal{U}$ and for almost all $t \in \mathbb{R}$ it holds that

$$\tfrac{d}{dt}\varphi_s\left(\tfrac{t}{s}, x, \tilde{u}\right) = s \cdot F\left(\varphi_s\left(\tfrac{t}{s}, x, \tilde{u}\right), \tilde{u}\left(\tfrac{t}{s}\right)\right)\tfrac{1}{s} = F\left(\varphi_s\left(\tfrac{t}{s}, x, \tilde{u}\right), u(t)\right).$$

Since $\varphi_s(0, x, \tilde{u}) = x$, by uniqueness of solutions, we obtain

$$\varphi_s\left(\tfrac{t}{s}, x, \tilde{u}\right) = \varphi(t, x, u) \quad \text{for all } t \in \mathbb{R}.$$

This implies that every (T,ε)-spanning set for (K,Q) with respect to system (1.7) yields a $(\frac{T}{s},\varepsilon)$-spanning set for (K,Q) with respect to system (2.8) with the same number of elements, and vice versa. Hence, $r_{\mathrm{inv}}(T,\varepsilon,K,Q;F) = r_{\mathrm{inv}}(\frac{T}{s},\varepsilon,K,Q;sF)$. We obtain

$$\begin{aligned} h_{\mathrm{inv}}(\varepsilon,K,Q;F) &= \limsup_{T\to\infty} \frac{1}{T} \ln r_{\mathrm{inv}}(T,\varepsilon,K,Q;F) \\ &= \limsup_{T\to\infty} \frac{1}{T} \ln r_{\mathrm{inv}}\left(\tfrac{T}{s},\varepsilon,K,Q;sF\right) \\ &= \frac{1}{s} \limsup_{T\to\infty} \frac{s}{T} \ln r_{\mathrm{inv}}\left(\tfrac{T}{s},\varepsilon,K,Q;sF\right) = \frac{1}{s} h_{\mathrm{inv}}(\varepsilon,K,Q;sF). \end{aligned}$$

For $\varepsilon \searrow 0$ it follows that $h_{\mathrm{inv}}(K,Q;sF) = s \cdot h_{\mathrm{inv}}(K,Q;F)$. The corresponding assertion for $h^*_{\mathrm{inv}}(K,Q;sF)$ is proved analogously. □

In order to prove the next proposition, we need a technical lemma:

2.2.3 Lemma:
For any functions $f_1,\dots,f_N : \mathbb{R}_0^+ \to (0,\infty)$ $(N \in \mathbb{N})$ it holds that

$$\limsup_{T\to\infty} \frac{1}{T} \ln \sum_{i=1}^{N} f_i(T) \leq \max_{i=1,\dots,N} \limsup_{T\to\infty} \frac{1}{T} \ln f_i(T).$$

Proof:
For brevity we write

$$\lambda(f) := \limsup_{T\to\infty} \frac{1}{T} \ln f(T)$$

for any function $f : \mathbb{R}_0^+ \to (0,\infty)$. We define $g : \mathbb{R}_0^+ \to (0,\infty)$ by

$$g(T) := \max_{i=1,\dots,N} f_i(T) \quad \text{for all } T \in \mathbb{R}_0^+.$$

Then

$$\lambda\left(\sum_{i=1}^{N} f_i\right) \leq \lambda(Ng) = \limsup_{T\to\infty} \frac{1}{T} (\ln N + \ln g(T)) = \lambda(g).$$

Thus, it suffices to show that $\lambda(g) \leq \max_{i=1,\dots,N} \lambda(f_i)$. To this end, let $(T_k)_{k\in\mathbb{N}}$, $T_k \in \mathbb{R}_0^+$, be a sequence with $T_k \to \infty$ and

$$\lambda(g) = \lim_{k\to\infty} \frac{1}{T_k} \ln \max_{i=1,\dots,N} f_i(T_k).$$

Obviously, there exists $i_0 \in \{1,\dots,N\}$ such that $f_{i_0}(T_k) = \max_{i=1,\dots,N} f_i(T_k)$ for infinitely many $k \in \mathbb{N}$. Let $(T_{n_k})_{k\in\mathbb{N}}$ be a corresponding subsequence. Then

$$\lambda(g) = \lim_{k\to\infty} \frac{1}{T_{n_k}} \ln f_{i_0}(T_{n_k}) \leq \lambda(f_{i_0}) \leq \max_{i=1,\dots,N} \lambda(f_i),$$

which finishes the proof. □

2.2.4 Proposition:
Consider control system (1.7). Let $Q \subset M$ be a compact controlled invariant set, and $K \subset Q$ compact. Assume that $K = \bigcup_{i=1}^{N} K_i$ with finitely many compact sets $K_1, \ldots, K_N$. Then

$$\boxed{h_{\text{inv}}^*(K,Q) = \max_{i=1,\ldots,N} h_{\text{inv}}^*(K_i,Q), \quad h_{\text{inv}}(K,Q) = \max_{i=1,\ldots,N} h_{\text{inv}}(K_i,Q).}$$

Proof:
If $\mathcal{S}$ is a minimal (T,ε)-spanning set for (K,Q), then $\mathcal{S}$ is also (T,ε)-spanning for (K_i,Q), $i = 1,\ldots,N$. Thus, we obtain

$$r_{\text{inv}}(T,\varepsilon,K_i,Q) \leq r_{\text{inv}}(T,\varepsilon,K,Q) \Rightarrow \max_{i=1,\ldots,N} h_{\text{inv}}(K_i,Q) \leq h_{\text{inv}}(K,Q).$$

On the other hand, if $\mathcal{S}_i$ is a minimal (T,ε)-spanning set for (K_i,Q), $i = 1,\ldots,N$, then $\mathcal{S} := \bigcup_{i=1}^{N} \mathcal{S}_i$ is (T,ε)-spanning for (K,Q), which implies

$$r_{\text{inv}}(T,\varepsilon,K,Q) \leq \#\mathcal{S} \leq \sum_{i=1}^{N} \#\mathcal{S}_i = \sum_{i=1}^{N} r_{\text{inv}}(T,\varepsilon,K_i,Q).$$

By Lemma 2.2.3 we obtain

$$h_{\text{inv}}(\varepsilon,K,Q) \leq \limsup_{T\to\infty} \frac{1}{T} \ln \sum_{i=1}^{N} r_{\text{inv}}(T,\varepsilon,K_i,Q) \leq \max_{i=1,\ldots,N} h_{\text{inv}}(\varepsilon,K_i,Q),$$

which yields the result for $\varepsilon \searrow 0$. The corresponding assertion for the strict invariance entropy is proved analogously. $\square$

2.2.5 Remark:
In Section 3.2, we will show, by counterexample, that Proposition 2.2.4 does not hold true for arbitrary countable coverings of K (see Remark 3.2.11).

Another natural question is under which transformations the invariance entropy is preserved. It turns out that a coordinate change in the state space, as described in the following definition, is the appropriate kind of transformation.

2.2.6 Definition (Topological Conjugacy):
Consider two control systems

$$\dot{x}(t) = F(x(t),u(t)), \quad u \in \mathcal{U}, \tag{2.9}$$

and

$$\dot{y}(t) = G(y(t),v(t)), \quad v \in \mathcal{V}, \tag{2.10}$$

on smooth manifolds M and N with control ranges U and V and (globally defined) cocycles φ and ψ, respectively. Let $\mathcal{U}$ and $\mathcal{V}$ denote the corresponding families of admissible control functions, and let $\pi : M \to N$, $h : \mathcal{U} \to \mathcal{V}$ be maps such that π is continuous and the following identity holds:

$$\pi(\varphi(t,x,u)) = \psi(t,\pi(x),h(u)) \quad \textit{for all } (t,x,u) \in \mathbb{R} \times M \times \mathcal{U}. \tag{2.11}$$

Then we say that system (2.9) is **topologically semiconjugate** *to system (2.10), and we call the pair (π, h) a* **topological semiconjugacy**. *If π is a homeomorphism and h is invertible, then the systems are called* **topologically conjugate** *and (π, h) is called a* **topological conjugacy** *from system (2.9) to system (2.10).*

2.2.7 Remark:

Note that what we call topological conjugacy is usually called *state equivalence* in control theory, provided that the mapping π is a diffeomorphism and $h = \mathrm{id}_{\mathcal{U}}$. See, e.g., Agrachev & Sachkov [3, Definition 5.23, p. 77] or Jakubczyk [31].

2.2.8 Proposition:

Consider the control systems (2.9) and (2.10). Let $Q \subset M$ be a compact controlled invariant set with respect to system (2.9), and let $K \subset Q$ be compact. Then, if system (2.9) is topologically semiconjugate to system (2.10) with a semiconjugacy (π, h), the set $\pi(Q) \subset N$ is controlled invariant with respect to system (2.10) and

$$\boxed{h^*_{\mathrm{inv}}(\pi(K), \pi(Q); G) \leq h^*_{\mathrm{inv}}(K, Q; F), \quad h_{\mathrm{inv}}(\pi(K), \pi(Q); G) \leq h_{\mathrm{inv}}(K, Q; F).}$$

Equation (2.11) in particular holds if $\pi : M \to N$ is a C^1-map and $H : U \to V$ is a continuous map such that

$$D\pi_x F(x, u) = G(\pi(x), H(u)) \quad \text{for all } (x, u) \in M \times U. \tag{2.12}$$

Proof:

By the assumptions it is clear that $\pi(K)$ and $\pi(Q)$ are nonvoid compact subsets of N with $\pi(K) \subset \pi(Q)$. Equation (2.11) implies controlled invariance of $\pi(Q)$ with respect to system (2.10): If $y \in \pi(Q)$, then there exists $x \in Q$ with $\pi(x) = y$. Let $u \in \mathcal{U}$ be a control function with $\varphi(\mathbb{R}^+_0, x, u) \subset Q$. Then it follows that

$$\psi(t, y, h(u)) = \psi(t, \pi(x), h(u)) \overset{(2.11)}{=} \pi(\varphi(t, x, u)) \in \pi(Q) \quad \text{for all } t \geq 0.$$

Now let $T, \varepsilon > 0$. Since π is uniformly continuous on the compact set Q, there exists $\delta > 0$ with $\pi(N_\delta(Q)) \subset N_\varepsilon(\pi(Q))$. Let $\mathcal{S} \subset \mathcal{U}$ be a minimal (T, δ)-spanning set for (K, Q) and define $\tilde{\mathcal{S}} := h(\mathcal{S})$. For any $y \in \pi(K)$ there exists $x \in K$ with $\pi(x) = y$. Let $u \in \mathcal{S}$ such that $\varphi([0, T], x, u) \subset N_\delta(Q)$. Then $h(u) \in \tilde{\mathcal{S}}$ and $\psi([0, T], \pi(x), h(u)) \subset \pi(N_\delta(Q)) \subset N_\varepsilon(\pi(Q))$. This shows that $\tilde{\mathcal{S}}$ is (T, ε)-spanning for $(\pi(K), \pi(Q))$. Consequently,

$$h_{\mathrm{inv}}(\varepsilon, \pi(K), \pi(Q); G) \leq h_{\mathrm{inv}}(\delta, K, Q; F) \leq h_{\mathrm{inv}}(K, Q; F).$$

For $\varepsilon \searrow 0$ we obtain $h_{\mathrm{inv}}(\pi(K), \pi(Q); G) \leq h_{\mathrm{inv}}(K, Q; F)$. It is even easier to see that the same inequality holds for the strict invariance entropy.

In order to see that the second assertion holds, recall that the solution $\varphi(\cdot, x, u) : \mathbb{R} \to M$ is the unique locally absolutely continuous curve with $\varphi(0, x, u) = x$ and

$$\frac{d}{dt}\varphi(t, x, u) = F(\varphi(t, x, u), u(t)) \quad \text{for almost all } t \in \mathbb{R}.$$

By the chain rule we obtain for almost all $t \in \mathbb{R}$ that

$$\begin{aligned}\frac{d}{dt}\pi(\varphi(t,x,u)) &= D\pi_{\varphi(t,x,u)}\frac{d}{dt}\varphi(t,x,u)\\ &= D\pi_{\varphi(t,x,u)}F(\varphi(t,x,u),u(t)) \overset{(2.12)}{=} G(\pi(\varphi(t,x,u)),H(u(t))).\end{aligned}$$

It follows that $\pi(\varphi(\cdot,x,u)) : \mathbb{R} \to N$ is a locally absolutely continuous curve on N with $\pi(\varphi(0,x,u)) = \pi(x)$, which satisfies the differential equation $\dot{y}(t) = G(y(t),H(u(t)))$ almost everywhere ($\pi(\varphi(\cdot,x,u))$ is locally absolutely continuous, since π is a C^1-map and hence locally Lipschitz continuous.) Let $h : \mathcal{U} \to \mathcal{V}$ be defined by $h(u)(t) := H(u(t))$ for all $u \in \mathcal{U}$ and $t \in \mathbb{R}$. Since H is continuous, $H \circ u$ is measurable for all $u \in \mathcal{U}$, and thus h is well-defined. Then, by uniqueness of solutions, it follows that $\pi(\varphi(t,x,u)) = \psi(t,\pi(x),h(u))$ for all $(t,x,u) \in \mathbb{R} \times M \times \mathcal{U}$. □

2.2.9 Remarks:

- If the pair (π,h) in Proposition 2.2.8 is a topological conjugacy, it follows that $h_{\text{inv}}^{(*)}(K,Q;F) = h_{\text{inv}}^{(*)}(\pi(K),\pi(Q);G)$, since then (2.11) implies

  $$\pi^{-1}(\psi(t,y,v)) = \varphi(t,\pi^{-1}(x),h^{-1}(v)) \quad \text{for all } (t,y,v) \in \mathbb{R} \times N \times \mathcal{V}.$$

  Hence, the (strict) invariance entropy is indeed preserved under topological conjugation.
- To obtain the result of Proposition 2.2.8 it would be sufficient to require the conjugacy identity (2.11) only for positive times $t \geq 0$.
- In Jakubczyk [31] and Agrachev & Sachkov [3, Section 5.7] necessary and sufficient conditions for smooth conjugacy (state equivalence) of control systems are analyzed.

The next proposition shows that for the computation of the invariance entropy it is sufficient to consider the system at times which are integer multiples of some fixed time step $\tau > 0$.

2.2.10 Proposition:
Consider control system (1.7) *and let $K, Q \subset M$ be compact sets with $K \subset Q$ and Q being controlled invariant. Then, for all $\varepsilon > 0$ and $\tau > 0$*

$$\boxed{h_{\text{inv}}(\varepsilon,K,Q) = \limsup_{n\to\infty}\frac{1}{n\tau}\ln r_{\text{inv}}(n\tau,\varepsilon,K,Q)} \tag{2.13}$$

and

$$\boxed{h_{\text{inv}}^*(K,Q) = \limsup_{n\to\infty}\frac{1}{n\tau}\ln r_{\text{inv}}^*(n\tau,K,Q).} \tag{2.14}$$

Proof:
Obviously, the left-hand side of (2.13) is not less than the right-hand side. In order to show the reverse, let $(T_k)_{k\in\mathbb{N}}$, $T_k > 0$, be an arbitrary sequence

converging to ∞. Then for every $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that $n_k\tau \leq T_k \leq (n_k+1)\tau$, and $n_k \to \infty$ for $k \to \infty$. By Proposition 2.1.8(i) we have

$$r_{\text{inv}}(T_k,\varepsilon,K,Q) \leq r_{\text{inv}}((n_k+1)\tau,\varepsilon,K,Q)$$

and consequently

$$\frac{1}{T_k}\ln r_{\text{inv}}(T_k,\varepsilon,K,Q) \leq \frac{1}{n_k\tau}\ln r_{\text{inv}}((n_k+1)\tau,\varepsilon,K,Q).$$

This yields

$$\limsup_{k\to\infty}\frac{1}{T_k}\ln r_{\text{inv}}(T_k,\varepsilon,K,Q) \leq \limsup_{k\to\infty}\frac{1}{n_k\tau}\ln r_{\text{inv}}((n_k+1)\tau,\varepsilon,K,Q).$$

Since

$$\frac{1}{n_k\tau}\ln r_{\text{inv}}((n_k+1)\tau,\varepsilon,K,Q) = \frac{n_k+1}{n_k}\frac{1}{(n_k+1)\tau}\ln r_{\text{inv}}((n_k+1)\tau,\varepsilon,K,Q)$$

and $\frac{n_k+1}{n_k} \to 1$ for $k \to \infty$, we obtain

$$\begin{aligned}\limsup_{k\to\infty}\frac{1}{T_k}\ln r_{\text{inv}}(T_k,\varepsilon,K,Q) &\leq \limsup_{k\to\infty}\frac{1}{n_k\tau}\ln r_{\text{inv}}(n_k\tau,\varepsilon,K,Q)\\ &\leq \limsup_{n\to\infty}\frac{1}{n\tau}\ln r_{\text{inv}}(n\tau,\varepsilon,K,Q).\end{aligned}$$

Formula (2.14) is proved analogously. □

Next, we define (strict) invariance entropy also for certain noncompact subsets of the state space.

2.2.11 Definition (Invariance Entropy for Noncompact Sets): *Consider control system (1.7) and assume that N is another smooth manifold of dimension $\dim(N) \leq d$ and that $\pi : M \to N$ is a surjective C^1-submersion. Suppose that the control system (1.7) can be projected onto N via π, which means that*

$$G : N\times\mathbb{R}^m \to TN,\ \ G(\pi(x),u) := D\pi_x F(x,u) \ \ \text{for all } (x,u) \in M\times\mathbb{R}^m, \quad (2.15)$$

is a well-defined C^1-mapping. Consider the control system

$$\dot{y}(t) = G(y(t),u(t)),\quad u \in \mathcal{U}. \qquad (2.16)$$

Let $K,Q \subset M$ be closed with $K \subset Q$ and Q being controlled invariant. Suppose that $\pi(K)$ and $\pi(Q)$ are compact subsets of N with $K = \pi^{-1}(\pi(K))$ and $Q = \pi^{-1}(\pi(Q))$. Then we define

$$h^*_{\text{inv,nc}}(K,Q;\pi) := h^*_{\text{inv}}(\pi(K),\pi(Q)),\quad h_{\text{inv,nc}}(K,Q;\pi) := h_{\text{inv}}(\pi(K),\pi(Q)), \qquad (2.17)$$

where the invariance entropy on the right-hand side is measured with respect to system (2.16).

2.2.12 Remark:
Definition 2.2.11 makes sense, since $K \subset Q$ implies $\pi(K) \subset \pi(Q)$ and $\pi(Q)$ is controlled invariant with respect to system (2.16), which is shown as in the proof of Proposition 2.2.8. Now assume that $\mathcal{S}$ is a T-spanning set for $(\pi(K), \pi(Q))$ with respect to system (2.16). Let $x \in K$ be chosen arbitrarily. Then there exists $u \in \mathcal{S}$ such that $\psi([0,T], \pi(x), u) \subset \pi(Q)$, where ψ denotes the cocycle corresponding to system (2.16). Since π maps solutions of (1.7) to solutions of (2.16) (see the proof of Proposition 2.2.8), we have

$$\varphi([0,T], x, u) \subset \pi^{-1}(\psi([0,T], \pi(x), u)) \subset \pi^{-1}(\pi(Q)) = Q.$$

This shows that Definition 2.2.11 is reasonable.

2.2.13 Example:
Consider a bilinear control system

$$\dot{x}(t) = \underbrace{\left[A_0 + \sum_{i=1}^{m} u_i(t) A_i\right]}_{=:A(u(t))} x(t), \quad u \in \mathcal{U}, \tag{2.18}$$

where $A_0, A_1, \ldots, A_m \in \mathbb{R}^{d \times d}$. We regard (2.18) as a control system on $\mathbb{R}^d \backslash \{0\}$. Let

$$\pi : \mathbb{R}^d \backslash \{0\} \to S^{d-1}, \quad \pi(x) = \frac{x}{\|x\|},$$

be the radial projection onto the sphere $S^{d-1} = \{s \in \mathbb{R}^d \mid \|s\| = 1\}$. It can easily be verified that π is a C^1-submersion with derivative

$$D\pi(x) = \frac{1}{\|x\|}\left(I - \frac{xx^T}{\|x\|^2}\right).$$

The bilinear system (2.18) can be projected to S^{d-1} via π and the right-hand side of the projected system is given by

$$G(s,u) = (A(u) - s^T A(u) s I)s, \quad S^{d-1} \times \mathbb{R}^m \to TS^{d-1},$$

which follows from

$$\begin{aligned} D\pi(x)A(u)x &= \frac{1}{\|x\|}\left(I - \frac{xx^T}{\|x\|^2}\right) A(u)x \\ &= \left(A(u) - \frac{xx^T A(u)}{\|x\|^2}\right)\frac{x}{\|x\|} \\ &= \left(A(u)x - \frac{x(x^T A(u) x)}{\|x\|^2}\right)\frac{1}{\|x\|} \\ &= \left(A(u) - \frac{x^T A(u) x}{\|x\|^2} I\right)\frac{x}{\|x\|} \\ &= \left(A(u) - \pi(x)^T A(u) \pi(x) I\right)\pi(x) = G(\pi(x), u). \end{aligned}$$

It is easy to see that it is also possible to project the bilinear system (2.18) to the $(d-1)$-dimensional real projective space $\mathbb{P}^{d-1}$, defined as the quotient space of $\mathbb{R}^d\backslash\{0\}$ under the equivalence relation

$$x_1 \sim x_2 \quad :\Leftrightarrow \quad \exists \lambda \in \mathbb{R}\backslash\{0\}: \; x_2 = \lambda x_1,$$

which identifies points that lie on the same line through the origin. $\lozenge$

In this section, we have proved elementary properties of the (strict) invariance entropy, some of which are also known to be valid in a similar form for the topological entropy of a flow. The following table provides a comparison of the corresponding properties.

Topological Entropy	Invariance Entropy
$\dot{x}(t) = s \cdot f(x(t))$ with flow Φ^s $\Rightarrow h_{\mathrm{top}}(\Phi^s) = s \cdot h_{\mathrm{top}}(\Phi)$	$\dot{x}(t) = s \cdot F(x(t), u(t)), \quad u \in \mathcal{U}$ $\Rightarrow h_{\mathrm{inv},s}(K,Q) = s \cdot h_{\mathrm{inv}}(K,Q)$
$\pi \circ \Phi_t = \Psi_t \circ \pi$ $\Rightarrow h_{\mathrm{top}}(\Psi) \leq h_{\mathrm{top}}(\Phi)$	$\pi(\varphi(t,x,u)) = \psi(t,\pi(x),h(u))$ $\Rightarrow h_{\mathrm{inv}}(\pi(K),\pi(Q)) \leq h_{\mathrm{inv}}(K,Q)$
$K = K_1 \dot{\cup} \cdots \dot{\cup} K_N$, $\Phi_t(K_i) \subset K_i$ $\Rightarrow h_{\mathrm{top}}(K,\Phi) = \max_i h_{\mathrm{top}}(K_i,\Phi)$	$K = \bigcup_{i=1}^N K_i$ $\Rightarrow h_{\mathrm{inv}}(K,Q) = \max_i h_{\mathrm{inv}}(K_i,Q)$

For the properties of the topological entropy stated above see Bowen [10, Proposition 21] and Adler & Konheim & McAndrew [2, Theorem 1 and Theorem 4], and note that the topological entropy of a flow equals that of its time-one-map.

2.2.14 Open Question:
Does $h_{\mathrm{inv}}^{(*)}(K,Q)$ depend continuously on K and/or on Q in some sense and under some condition?

2.3 Isolated Sets and Inner Control Sets

Now we examine how the behavior of a control-affine system in a neighborhood of a controlled invariant set Q is related to the invariance entropy of Q. We consider two opposite situations. First we assume that the set Q is isolated in the sense that any trajectory, which does not leave a fixed neighborhood of Q, is already contained in Q. In this situation, it turns out that the invariance entropy is given by $h_{\mathrm{inv}}(\varepsilon,K,Q)$ for small ε and hence taking the limit $\varepsilon \searrow 0$ is unnecessary. Then we consider the situation that the system is controllable in a neighborhood of Q (in a sense which will be made precise). Here we will see that the limes superior in the definition of the invariance entropy can be replaced by a limes inferior. Both of these results are inspired by analog results for the topological entropy: For expansive homeomorphisms it is also not necessary to take the limit $\varepsilon \searrow 0$ in the definition of topological entropy via separated and hence also via spanning sets (see Katok & Hasselblatt [33, Corollary 3.2.13, p. 126]). Replacing the limes superior by a limes inferior in the definition via spanning sets is always possible (see Mane [37, Proposition 7.1, p. 237]).

First we prove a useful lemma.

2.3.1 Lemma:
Let D be a control set of the control-affine system (1.34)*. Then* $\operatorname{cl} D$ *is controlled invariant.*

Proof:
Let $x \in \operatorname{cl} D$ be chosen arbitrarily. Then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in D$ for all $n \in \mathbb{N}$ and $x_n \to x$. Since D is controlled invariant, for every $n \in \mathbb{N}$ there is $u_n \in \mathcal{U}$ with $\varphi(\mathbb{R}_0^+, x_n, u_n) \subset D \subset \operatorname{cl} D$. Since $\mathcal{U}$ is compact with the weak*-topology, we may assume that $u_n \to u$ for some $u \in \mathcal{U}$. By continuity of φ we have $\varphi(t, x_n, u_n) \to \varphi(t, x, u)$ for all $t \geq 0$, which implies $\varphi(\mathbb{R}_0^+, x, u) \subset \operatorname{cl} D$. Hence, $\operatorname{cl} D$ is controlled invariant. □

Next, we give a definition of isolated sets and inner control sets.

2.3.2 Definition (Isolated Sets and Inner Control Sets):
Consider the control-affine system (1.34) *with compact and convex control range $U \subset \mathbb{R}^m$ and let A be a subset of M.*

(i) *A is called* **isolated** *if there exists $\delta_0 > 0$ such that for all $(x, u) \in \operatorname{cl} N_{\delta_0}(A) \times \mathcal{U}$ the implication*

$$\varphi(\mathbb{R}, x, u) \subset \operatorname{cl} N_{\delta_0}(A) \quad \Rightarrow \quad \varphi(\mathbb{R}, x, u) \subset A \tag{2.19}$$

holds.

(ii) *A is called* **isolated in forward time** *if there exists $\delta_0 > 0$ such that for all $(x, u) \in \operatorname{cl} N_{\delta_0}(A) \times \mathcal{U}$ the implication*

$$\varphi(\mathbb{R}_0^+, x, u) \subset \operatorname{cl} N_{\delta_0}(A) \quad \Rightarrow \quad \varphi(\mathbb{R}_0^+, x, u) \subset A \tag{2.20}$$

holds.

(iii) *A is called an* **inner control set** *if there exists a decreasing family of compact and convex sets $\{U_\rho\}_{\rho\in[0,1]}$ in $\mathbb{R}^m$ (i.e., $U_{\rho_2} \subset U_{\rho_1}$ for $\rho_2 > \rho_1$), such that for every $\rho \in [0, 1]$ the control-affine system* (1.34) *with control range U_ρ (instead of U) has a control set D_ρ with nonvoid interior and compact closure, and the following conditions are satisfied:*
   (a) *$U = U_0$ and $A = D_1$.*
   (b) *$\operatorname{cl} D_{\rho_2} \subset \operatorname{int} D_{\rho_1}$ whenever $\rho_1 < \rho_2$.*
   (c) *For every neighborhood W of $\operatorname{cl} A$ there is $\rho \in [0, 1)$ with $\operatorname{cl} D_\rho \subset W$.*

In the next proposition, we assume that the set Q is isolated in forward time.

2.3.3 Proposition:
Consider the control-affine system (1.34). Let $Q \subset M$ be compact, controlled invariant and isolated in forward time with constant δ_0. Then for every compact set $K \subset Q$ it holds that

$$\boxed{h_{\mathrm{inv}}(K,Q) = h_{\mathrm{inv}}(\varepsilon,K,Q) \quad \textit{for all } \varepsilon \in (0,\delta_0].}$$

Proof:
We may assume that δ_0 is small enough that $\operatorname{cl} N_{\delta_0}(Q)$ is compact, since the assumption (2.20) is also satisfied for smaller δ_0 (see Lemma A.3.2). Then first we show the following:

$$\begin{aligned}&\forall \rho > 0 : \;\; \forall \varepsilon \in (0,\delta_0] : \; \exists n \in \mathbb{N} : \; \forall (x,u) \in \operatorname{cl} N_{\delta_0}(Q) \times \mathcal{U} : \\ &\qquad\qquad \max_{t\in[0,n]} \operatorname{dist}(\varphi(t,x,u),Q) \leq \varepsilon \quad \Rightarrow \quad \operatorname{dist}(x,Q) < \rho.\end{aligned}$$

To this end, assume that the opposite is true:

$$\begin{aligned}&\exists \rho > 0 : \;\; \exists \varepsilon \in (0,\delta_0] : \; \forall n \in \mathbb{N} : \; \exists (x_n,u_n) \in \operatorname{cl} N_{\delta_0}(Q) \times \mathcal{U} : \\ &\qquad\qquad \max_{t\in[0,n]} \operatorname{dist}(\varphi(t,x_n,u_n),Q) \leq \varepsilon \quad \text{and} \quad \operatorname{dist}(x_n,Q) \geq \rho.\end{aligned}$$

By compactness of $\operatorname{cl} N_{\delta_0}(Q)$ and $\mathcal{U}$ (endowed with the weak*-topology) we may assume that $(x_n,u_n) \to (x,u) \in \operatorname{cl} N_{\delta_0}(Q) \times \mathcal{U}$. By continuity of $\operatorname{dist}(\cdot,Q)$ (see Lemma A.3.1) we obtain

$$\operatorname{dist}(x,Q) = \lim_{n\to\infty} \operatorname{dist}(x_n,Q) \geq \rho \quad \Rightarrow \quad x \notin Q.$$

For arbitrary $t_0 \geq 0$ we have

$$\begin{aligned}\operatorname{dist}(\varphi(t_0,x,u),Q) &= \lim_{n\to\infty} \operatorname{dist}(\varphi(t_0,x_n,u_n),Q) \\ &\leq \limsup_{n\to\infty} \underbrace{\max_{t\in[0,t_0]} \operatorname{dist}(\varphi(t,x_n,u_n),Q)}_{\leq \varepsilon \text{ for } n \geq t_0} \leq \varepsilon \leq \delta_0.\end{aligned}$$

Hence, $\varphi(\mathbb{R}_0^+,x,u) \subset \operatorname{cl} N_{\delta_0}(Q)$ which implies $\varphi(\mathbb{R}_0^+,x,u) \subset Q$ in contradiction to $x \notin Q$.

Now let $0 < \varepsilon_1 < \varepsilon_2 \leq \delta_0$. Then, by what we have shown, there exists $n \in \mathbb{N}$ such that for all $(x,u) \in \operatorname{cl} N_{\delta_0}(Q) \times \mathcal{U}$ it holds that

$$\max_{t\in[0,n]} \operatorname{dist}(\varphi(t,x,u),Q) \leq \varepsilon_2 \quad \Rightarrow \quad \operatorname{dist}(x,Q) < \varepsilon_1. \tag{2.21}$$

For arbitrary $T > 0$ let $\mathcal{S}$ be a minimal $(n+T,\varepsilon_2)$-spanning set for (K,Q). Pick $x \in K$. Then there exists $u_x \in \mathcal{S}$ with

$$\varphi\left([0,n+T],x,u_x\right) \subset N_{\varepsilon_2}(Q).$$

Let $s \in [0,T]$. Then

$$\max_{t\in[0,n]} \operatorname{dist}(\varphi(t,\varphi(s,x,u_x),\Theta_s u_x),Q) = \max_{t\in[0,n]} \operatorname{dist}(\varphi(t+s,x,u_x),Q) < \varepsilon_2.$$

Hence, by (2.21) we have

$$\operatorname{dist}(\varphi(s,x,u_x),Q) < \varepsilon_1 \quad \text{for all } s \in [0,T],$$

which implies that $\mathcal{S}$ is a (T,ε_1)-spanning set for (K,Q). Therefore,

$$r_{\text{inv}}(T,\varepsilon_1,K,Q) \leq r_{\text{inv}}(n+T,\varepsilon_2,K,Q) \quad \text{for all } T>0,$$

which immediately gives

$$h_{\text{inv}}(\varepsilon_1,K,Q) \leq h_{\text{inv}}(\varepsilon_2,K,Q).$$

Together with $h_{\text{inv}}(\varepsilon_2,K,Q) \leq h_{\text{inv}}(\varepsilon_1,K,Q)$ (see Proposition 2.1.8(ii)) this implies the result. □

2.3.4 Example:
Consider a linear control system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad u \in \mathcal{U},$$

with $A \in \mathbb{R}^{d\times d}$ and $B \in \mathbb{R}^{d\times m}$ such that (A,B) is controllable, the control range U is compact and convex with $0 \in \operatorname{int} U$, and all eigenvalues of A have positive real parts. Then there exists a unique open control set $D \subset \mathbb{R}^d$ with compact closure Q, which is given by $D = \mathcal{O}^-(0)$. To prove the latter, note that $0 \in \operatorname{int} D$ (see Colonius & Kliemann [16, Example 3.2.16, p. 61]) and hence $D = \operatorname{cl}\mathcal{O}^+(0) \cap \mathcal{O}^-(0)$ by Formula (1.33). Since $\operatorname{int} D \subset \mathcal{O}^+(0)$, we have

$$\varphi\left(\mathbb{R}_0^+, \operatorname{int} D, 0\right) = \bigcup_{t\geq 0} e^{At} \operatorname{int} D \subset \mathcal{O}^+(0),$$

and since e^{At} is expanding, this implies $\mathbb{R}^d \subset \mathcal{O}^+(0)$. In particular, there are constants $c,\alpha > 0$ such that

$$\|e^{At}x\| \geq ce^{\alpha t}\|x\| \quad \text{for all } t \geq 0 \text{ and } x \in \mathbb{R}^d, \tag{2.22}$$

which follows from Lemma A.3.8. Now let $x \in \mathbb{R}^d \backslash Q$ and $u \in \mathcal{U}$. Define

$$\beta := \operatorname{dist}(x,Q) > 0.$$

For given $\tau > 0$ define

$$y := -\int_0^\tau e^{-As}Bu(s)ds.$$

Then $y \in \mathcal{O}^-(0) = D$, since

$$\varphi(\tau,y,u) = e^{A\tau}\left(-\int_0^\tau e^{-As}Bu(s)ds\right) + \int_0^\tau e^{A(\tau-s)}Bu(s)ds = 0.$$

This implies

$$\begin{aligned} \|\varphi(\tau,x,u)\| &= \|\varphi(\tau,x,u) - \varphi(\tau,y,u)\| = \|e^{A\tau}(x-y)\| \\ &\overset{(2.22)}{\geq} ce^{\alpha\tau} \underbrace{\|x-y\|}_{\geq \operatorname{dist}(x,Q)} \geq ce^{\alpha\tau}\beta. \end{aligned}$$

Hence, $\varphi(\tau,x,u) \to \infty$ for $\tau \to \infty$, which implies (2.20). ◊

Now we consider sets which are isolated with respect to the whole time. The proof of the following proposition is completely analogously to that of Proposition 2.3.3.

2.3.5 Proposition:
Consider the control-affine system (1.34). Let $Q \subset M$ be compact, controlled invariant both in forward and in backward time and isolated with constant δ_0. Then for every compact set $K \subset Q$ it holds that

$$\boxed{h_{\mathrm{inv}}(K,Q) = h_{\mathrm{inv}}(\varepsilon,K,Q) \quad \textit{for all } \varepsilon \in (0,\delta_0].}$$

Proof:
We may assume that δ_0 is small enough that $\mathrm{cl}\, N_{\delta_0}(Q)$ is compact, since the assumption (2.19) is also satisfied for smaller δ_0 (see Lemma A.3.2). Then first we show the following:

$$\begin{gathered}\forall \rho > 0: \ \forall \varepsilon \in (0,\delta_0]: \ \exists n \in \mathbb{N}: \ \forall (x,u) \in \mathrm{cl}\, N_{\delta_0}(Q) \times \mathcal{U}: \\ \max_{t \in [-n,n]} \mathrm{dist}(\varphi(t,x,u),Q) \leq \varepsilon \quad \Rightarrow \quad \mathrm{dist}(x,Q) < \rho.\end{gathered}$$

To this end, assume that the opposite is true:

$$\begin{gathered}\exists \rho > 0: \ \exists \varepsilon \in (0,\delta_0]: \ \forall n \in \mathbb{N}: \ \exists (x_n,u_n) \in \mathrm{cl}\, N_{\delta_0}(Q) \times \mathcal{U}: \\ \max_{t \in [-n,n]} \mathrm{dist}(\varphi(t,x_n,u_n),Q) \leq \varepsilon \quad \text{and} \quad \mathrm{dist}(x_n,Q) \geq \rho.\end{gathered}$$

By compactness of $\mathrm{cl}\, N_{\delta_0}(Q)$ and $\mathcal{U}$ (endowed with the weak*-topology) we may assume that $(x_n,u_n) \to (x,u) \in \mathrm{cl}\, N_{\delta_0}(Q) \times \mathcal{U}$. By continuity of $\mathrm{dist}(\cdot,Q)$ (see Lemma A.3.1) we obtain

$$\mathrm{dist}(x,Q) = \lim_{n\to\infty} \mathrm{dist}(x_n,Q) \geq \rho \quad \Rightarrow \quad x \notin Q.$$

For arbitrary $t_0 \in \mathbb{R}$ we obtain

$$\begin{aligned}\mathrm{dist}(\varphi(t_0,x,u),Q) &= \lim_{n\to\infty} \mathrm{dist}(\varphi(t_0,x_n,u_n),Q) \\ &\leq \limsup_{n\to\infty} \underbrace{\max_{t\in[-|t_0|,|t_0|]} \mathrm{dist}(\varphi(t,x_n,u_n),Q)}_{\leq \varepsilon \text{ for } n \geq |t_0|} \leq \varepsilon \leq \delta_0.\end{aligned}$$

Hence, $\varphi(\mathbb{R},x,u) \subset \mathrm{cl}\, N_{\delta_0}(Q)$, which implies $\varphi(\mathbb{R},x,u) \subset Q$ in contradiction to $x \notin Q$.

Now let $0 < \varepsilon_1 < \varepsilon_2 \leq \delta_0$. Then, by what we have shown, there exists $n \in \mathbb{N}$ such that for all $(x,u) \in \mathrm{cl}\, N_{\delta_0}(Q) \times \mathcal{U}$ it holds that

$$\max_{t\in[-n,n]} \mathrm{dist}(\varphi(t,x,u),Q) \leq \varepsilon_2 \quad \Rightarrow \quad \mathrm{dist}(x,Q) < \varepsilon_1. \tag{2.23}$$

For arbitrary $T > 0$ let $\mathcal{S}$ be a minimal $(n+T,\varepsilon_2)$-spanning set for (K,Q). Pick $x \in K$. Then there exists $u_x \in \mathcal{S}$ with

$$\varphi\left([0,n+T],x,u_x\right) \subset N_{\varepsilon_2}(Q).$$

Let $s \in [0, T]$. Then

$$\max_{t\in[0,n]} \operatorname{dist}(\varphi(t, \varphi(s, x, u_x), \Theta_s u_x), Q) = \max_{t\in[0,n]} \operatorname{dist}(\varphi(t + s, x, u_x), Q) < \varepsilon_2.$$

Since Q is controlled invariant in backward time by assumption, we can assume that $\varphi([-n, 0], x, u_x) \subset Q$ and hence

$$\max_{t\in[-n,0]} \operatorname{dist}(\varphi(t, \varphi(s, x, u_x), \Theta_s u_x), Q) = \max_{t\in[-n,0]} \operatorname{dist}(\varphi(t + s, x, u_x), Q) < \varepsilon_2.$$

Hence, by (2.23) we have

$$\operatorname{dist}(\varphi(s, x, u_x), Q) < \varepsilon_1 \quad \text{for all } s \in [0, T],$$

which implies that $\mathcal{S}$ is a (T, ε_1)-spanning set for (K, Q). Therefore,

$$r_{\text{inv}}(T, \varepsilon_1, K, Q) \le r_{\text{inv}}(n + T, \varepsilon_2, K, Q) \quad \text{for all } T > 0,$$

which immediately gives

$$h_{\text{inv}}(\varepsilon_1, K, Q) \le h_{\text{inv}}(\varepsilon_2, K, Q).$$

Together with $h_{\text{inv}}(\varepsilon_2, K, Q) \le h_{\text{inv}}(\varepsilon_1, K, Q)$ (see Proposition 2.1.8(ii)) this implies the result. □

2.3.6 Remark:
An example for a set which is controlled invariant in forward and in backward time is the closure of a control set D with nonvoid interior, provided that local accessibility holds on $\operatorname{cl} D$. This follows from Colonius & Kliemann [16, Lemma 3.2.22, p. 65], which says that there exists a unique control set D^* for the time-reversed system such that $\operatorname{int} D = \operatorname{int} D^*$. The closures of D and D^* coincide by Proposition 1.3.6, and hence, by Lemma 2.3.1, $\operatorname{cl} D$ is controlled invariant in backward time.

Now we turn to the case that Q is the closure of an inner control set.

2.3.7 Proposition:
Let Q be the closure of a relatively compact inner control set of the control-affine system (1.34)*. Then Q is controlled invariant and for every compact set $K \subset Q$ with nonvoid interior we have*

$$\boxed{h_{\text{inv}}(K, Q) = \lim_{\varepsilon \searrow 0} \liminf_{T\to\infty} \frac{1}{T} \ln r_{\text{inv}}(T, \varepsilon, K, Q).} \tag{2.24}$$

Proof:
Controlled invariance of Q follows from the fact that Q is the closure of the set D_1, which is a control set for the control range $U_1 \subset U$ (see Lemma 2.3.1). By the assumptions we can find a monotonically increasing sequence $(\rho_n)_{n\in\mathbb{N}}$ in $[0, 1)$ with $D_{\rho_n} \subset N_{1/n}(Q)$ for all $n \in \mathbb{N}$. Since $Q = \operatorname{cl} D_1 \subset \operatorname{int} D_{\rho_n}$ for all $n \in \mathbb{N}$, we can find a monotonically decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive real

numbers with $\lim_{n\to\infty}\varepsilon_n = 0$ such that $N_{\varepsilon_n}(Q) \subset D_{\rho_n}$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ it is possible to steer all points of $N_{\varepsilon_n}(Q)$ to K with finitely many control functions using the control range U_{ρ_n} and hence (by the no-return property, see Remark 1.3.9) not leaving the control set D_{ρ_n}, which is contained in $N_{1/n}(Q)$. Let α_n be the minimal number of control functions which are necessary to do so. Then for every $\tau > 0$ and $m \in \mathbb{N}$ we can construct an $(m\tau, \frac{1}{n})$-spanning set for (K,Q) with cardinality less than $\alpha_n^m r_{\mathrm{inv}}(\tau,\varepsilon,K,Q)^m$ for all $\varepsilon \in (0,\varepsilon_n]$ (by iterated concatenation of the control functions of a minimal (τ,ε)-spanning set for (K,Q) and the control functions used to steer the system from $N_{\varepsilon_n}(Q)$ to K). Hence, we obtain

$$r_{\mathrm{inv}}\left(m\tau, \tfrac{1}{n}, K, Q\right) \leq \alpha_n^m r_{\mathrm{inv}}(\tau,\varepsilon,K,Q)^m \quad \text{for all } m \in \mathbb{N},\ \tau > 0,\ 0 < \varepsilon \leq \varepsilon_n.$$

Using Proposition 2.2.10, this implies

$$\begin{aligned} h_{\mathrm{inv}}\left(\tfrac{1}{n}, K, Q\right) &= \limsup_{m\to\infty} \frac{1}{m\tau} \ln r_{\mathrm{inv}}\left(m\tau, \tfrac{1}{n}, K, Q\right) \\ &\leq \limsup_{m\to\infty} \frac{1}{\tau}\left(\ln \alpha_n + \ln r_{\mathrm{inv}}(\tau,\varepsilon,K,Q)\right) \\ &= \frac{1}{\tau}\ln\alpha_n + \frac{1}{\tau}\ln r_{\mathrm{inv}}(\tau,\varepsilon,K,Q). \end{aligned}$$

Therefore, we have

$$\begin{aligned} h_{\mathrm{inv}}\left(\tfrac{1}{n}, K, Q\right) &\leq \lim_{\varepsilon\searrow 0}\liminf_{\tau\to\infty}\left(\frac{1}{\tau}\ln\alpha_n + \frac{1}{\tau}\ln r_{\mathrm{inv}}(\tau,\varepsilon,K,Q)\right) \\ &= \lim_{\varepsilon\searrow 0}\liminf_{\tau\to\infty}\frac{1}{\tau}\ln r_{\mathrm{inv}}(\tau,\varepsilon,K,Q). \end{aligned}$$

This implies

$$h_{\mathrm{inv}}(K,Q) = \lim_{n\to\infty} h_{\mathrm{inv}}\left(\tfrac{1}{n}, K, Q\right) \leq \lim_{\varepsilon\searrow 0}\liminf_{\tau\to\infty}\frac{1}{\tau}\ln r_{\mathrm{inv}}(\tau,\varepsilon,K,Q),$$

which finishes the proof. □

2.3.8 Remark:
Note that from (2.24) it does not necessarily follow that the limit $\lim_{T\to\infty}\frac{1}{T}\ln r_{\mathrm{inv}}(T,\varepsilon,K,Q)$ exists for any $\varepsilon > 0$.

2.3.9 Open Question:
Is the isolation property (2.19) satisfied if Q is a chain control set?

Chapter 3

Estimates

In this chapter, we obtain lower and upper bounds for the invariance entropy, which can be computed directly from the right-hand side of the control system. The existence of upper bounds in particular shows finiteness of $h_{\mathrm{inv}}(K,Q)$. Since these bounds involve terms which depend on additional structures we have to impose on the state space—namely Riemannian metrics and volume forms—the question arises, how the bounds can be optimized by varying the corresponding structures. But this question seems to be quite difficult and except for some trivial examples one cannot hope to get tight bounds for the invariance entropy by doing such an optimization. Hence, we do not attempt to solve this problem. For one-dimensional linear systems and for projected bilinear systems on the unit sphere we compute the aforesaid bounds explicitly. Moreover, we introduce the notion of a *uniformly expanding* system, meaning a system, whose solutions expand a given metric on the state space uniformly for all control functions. We show that under the assumption that the set K has positive fractal dimension, $h_{\mathrm{inv}}(K,Q)$ is positive for a system which is uniformly expanding on Q. Finally, we are able to derive an explicit formula for the invariance entropy of one-dimensional linear systems.

3.1 Upper Bounds

In this section, we derive a metric-dependent upper bound for the invariance entropy $h_{\mathrm{inv}}(K,Q)$ in terms of the fractal dimension of K and the maximal eigenvalue of the symmetrized covariant derivative of the right-hand side. In particular, this proves finiteness of $h_{\mathrm{inv}}(K,Q)$. In order to do so, we first introduce another entropy-like quantity, which serves as a general upper bound for the invariance entropy.

3.1.1 Definition (Strong Invariance Entropy):
Consider control system (1.7)*. Let $Q \subset M$ be a compact controlled invariant set and let $K \subset Q$ be compact. Define the lift of Q by*

$$\mathcal{Q} := \left\{(x,u) \in M \times \mathcal{U} \mid \varphi(\mathbb{R}_0^+, x, u) \subset Q\right\}.$$

For $T, \varepsilon > 0$ a set $\mathcal{S}^+ \subset \mathcal{Q}$ is called **strongly (T, ε)-spanning** *for (K, Q) if*

$$\forall x \in K: \ \exists (y, u) \in \mathcal{S}^+ : \ \max_{t \in [0,T]} d(\varphi(t, x, u), \varphi(t, y, u)) < \varepsilon.$$

We let $r_{\mathrm{inv}}^+(T, \varepsilon, K, Q)$ denote the minimal number of elements in a strongly (T, ε)-spanning set. For every $\varepsilon > 0$ we define

$$h_{\mathrm{inv}}^+(\varepsilon, K, Q) := \limsup_{T \to \infty} \frac{1}{T} \ln r_{\mathrm{inv}}^+(T, \varepsilon, K, Q).$$

The **strong invariance entropy** *of (K, Q) is defined by*

$$h_{\mathrm{inv}}^+(K, Q) := \lim_{\varepsilon \searrow 0} h_{\mathrm{inv}}^+(\varepsilon, K, Q). \tag{3.1}$$

3.1.2 Remark:

Note that, in contrast to T-spanning and (T, ε)-spanning sets, the elements of strongly (T, ε)-spanning sets are pairs (x, u) of points and control functions.

3.1.3 Proposition:

(i) *$h_{\mathrm{inv}}^+(K, Q)$ is a well-defined number contained in $[0, \infty]$.*

(ii) *$h_{\mathrm{inv}}^+(K, Q)$ does not depend on the metric d.*

(iii) *The following estimates hold:*

$$\boxed{h_{\mathrm{inv}}(\varepsilon, K, Q) \leq h_{\mathrm{inv}}^+(\varepsilon, K, Q) \ \text{ and } \ h_{\mathrm{inv}}(K, Q) \leq h_{\mathrm{inv}}^+(K, Q).} \tag{3.2}$$

Proof:

(i) With the same arguments as used for the quantities $r_{\mathrm{inv}}(T, \varepsilon, K, Q)$ (compactness of K, continuous dependence on the initial value) one can show that the numbers $r_{\mathrm{inv}}^+(T, \varepsilon, K, Q)$ are finite. If $\varepsilon_1 < \varepsilon_2$, and $\mathcal{S}^+$ is a strongly (T, ε_1)-spanning set, then obviously it is also a strongly (T, ε_2)-spanning set, and hence $r_{\mathrm{inv}}^+(T, \varepsilon_2, K, Q) \leq r_{\mathrm{inv}}^+(T, \varepsilon_1, K, Q)$. This implies $h_{\mathrm{inv}}^+(\varepsilon_2, K, Q) \leq h_{\mathrm{inv}}^+(\varepsilon_1, K, Q)$ and thus the limit for $\varepsilon \searrow 0$ exists.

(ii) Let $\tilde{d}$ be another metric on M. Since Q is compact, the identity map $\mathrm{id} : (M, d) \to (M, \tilde{d})$ is uniformly continuous on Q, i.e.,

$$\forall \varepsilon > 0: \ \exists \delta > 0: \ \forall x \in Q: \ \forall y \in M: \ d(x, y) < \delta \Rightarrow \tilde{d}(x, y) < \varepsilon.$$

Now let $T, \varepsilon > 0$ be given and choose $\delta = \delta(\varepsilon)$ as above. Let $\mathcal{S}^+$ be a strongly (T, δ)-spanning set for (K, Q) with respect to the metric d, i.e.,

$$\forall x \in K: \ \exists (y, u) \in \mathcal{S}^+ : \ \forall t \in [0, T]: \ d(\varphi(t, x, u), \varphi(t, y, u)) < \delta.$$

Since $\varphi([0, T], y, u) \subset Q$, this implies

$$\forall x \in K: \ \exists (y, u) \in \mathcal{S}^+ : \ \forall t \in [0, T]: \ \tilde{d}(\varphi(t, x, u), \varphi(t, y, u)) < \varepsilon.$$

Consequently, $\mathcal{S}^+$ is strongly (T,ε)-spanning for (K,Q) with respect to $\tilde{d}$, which yields

$$r^+_{\mathrm{inv}}(T,\varepsilon,K,Q;\tilde{d}) \leq r^+_{\mathrm{inv}}(T,\delta,K,Q;d) \quad \text{for all } T>0.$$

Hence, $h^+_{\mathrm{inv}}(\varepsilon,K,Q;\tilde{d}) \leq h^+_{\mathrm{inv}}(\delta,K,Q;d)$. Letting $\varepsilon \to 0$ we obtain $h^+_{\mathrm{inv}}(K,Q;\tilde{d}) \leq h^+_{\mathrm{inv}}(K,Q;d)$. Changing the roles of d and $\tilde{d}$ yields the result.

(iii) Let $\mathcal{S}^+ = \{(y_1,u_1),\ldots,(y_n,u_n)\}$ be a minimal strongly (T,ε)-spanning set for (K,Q) and let $\mathcal{S} := \{u_1,\ldots,u_n\}$. We want to show that $\mathcal{S}$ is (T,ε)-spanning for (K,Q). To this end, pick $x \in K$ arbitrarily. Then there exists $i \in \{1,\ldots,n\}$ with $d(\varphi(t,x,u_i),\varphi(t,y_i,u_i)) < \varepsilon$ for all $t \in [0,T]$. Since $\mathcal{S}^+ \subset \mathcal{Q}$ we have $\varphi(t,y_i,u_i) \in Q$ for all $t \geq 0$ and this implies $\varphi([0,T],x,u_i) \subset N_\varepsilon(Q)$. Hence,

$$r_{\mathrm{inv}}(T,\varepsilon,K,Q) \leq \#\mathcal{S} \leq \#\mathcal{S}^+ = r^+_{\mathrm{inv}}(T,\varepsilon,K,Q).$$

Consequently, also $h_{\mathrm{inv}}(\varepsilon,K,Q) \leq h^+_{\mathrm{inv}}(\varepsilon,K,Q)$ and $h_{\mathrm{inv}}(K,Q) \leq h^+_{\mathrm{inv}}(K,Q)$.

□

Now we can prove the main theorem of this section.

3.1.4 Theorem (General Upper Bound):
Consider control system (1.7). Let $K,Q \subset M$ be compact sets with $K \subset Q$ and Q being controlled invariant. Let g be a Riemannian metric on M of class C^∞. Then the estimate

$$\boxed{h_{\mathrm{inv}}(K,Q) \leq \max\left\{0, \max_{(x,u)\in Q\times U} \lambda_{\max}(S\nabla F_u(x))\right\} \cdot \dim_F(K)} \tag{3.3}$$

holds, where $\lambda_{\max}(\cdot)$ denotes the maximal eigenvalue, $S\nabla\cdot$ the symmetrized covariant derivative of a vector field and $\dim_F(\cdot)$ the fractal dimension.

Proof:
The proof is subdivided into three parts.

Step 1: Let $\varepsilon > 0$ be chosen arbitrarily but small enough such that $\mathrm{cl}\, N_{2\varepsilon}(Q)$ is compact and for all $x \in Q$ the Riemannian exponential function $\exp_x$ is defined on the ball $B_\varepsilon(0) \subset T_xM$. By compactness of Q both is possible. For the first see Lemma A.3.2, and for the second Gallot & Hulin & Lafontaine [22, Corollary 2.89, p. 82].[1] By Lemma A.3.3 there exists a cut-off function $\theta : M \to [0,1]$ of class C^1 such that

$$\theta(x) \equiv 1 \text{ on } \mathrm{cl}\, N_\varepsilon(Q) \quad \text{and} \quad \theta(x) \equiv 0 \text{ on } M\backslash N_{2\varepsilon}(Q).$$

[1] This corollary states in particular that for every point x on a Riemannian manifold (M,g) there exists a neighborhood U of x and $\varepsilon > 0$ such that for all $y \in U$ the map $\exp_y$ is defined on $B_\varepsilon(0) \subset T_yM$.

We define a C^1-mapping $\tilde{F} : M \times \mathbb{R}^m \to TM$ by

$$\tilde{F}(x,u) := \theta(x)F(x,u) \quad \text{for all } (x,u) \in M \times \mathbb{R}^m,$$

and consider the control system

$$\dot{x}(t) = \tilde{F}(x(t), u(t)), \quad u \in \mathcal{U}. \tag{3.4}$$

The corresponding cocycle is denoted by $\tilde{\varphi}$. Note that by Corollary 1.2.6 all maximal solutions of (3.4) are defined on $\mathbb{R}$. By definition of $\tilde{F}$ we have

$$\varphi(t,x,u) = \tilde{\varphi}(t,x,u) \text{ whenever } \varphi([0,t],x,u) \subset \operatorname{cl} N_\varepsilon(Q) \tag{3.5}$$

for all $(t,x,u) \in \mathbb{R}_0^+ \times M \times \mathcal{U}$. In particular, this implies that Q is also controlled invariant with respect to system (3.4). Now we define for every $\tau > 0$ the set

$$\mathcal{D}(\tau) := [0,\tau] \times \operatorname{cl} N_\varepsilon(Q) \times \mathcal{U}$$

and the number

$$L_\varepsilon(\tau) := \sup_{(t,x,u) \in \mathcal{D}(\tau)} \left\| D\tilde{\varphi}_{t,u}(x) \right\|, \quad L_\varepsilon := L_\varepsilon(1), \tag{3.6}$$

where $\|\cdot\|$ denotes the operator norm induced by the Riemannian metric. Since $\tilde{\varphi}_{0,u}(x) \equiv x$ on $M \times \mathcal{U}$, we have

$$L_\varepsilon(\tau) \geq \sup_{(x,u) \in \operatorname{cl} N_\varepsilon(Q) \times \mathcal{U}} \|D\tilde{\varphi}_{0,u}(x)\| = \sup_{x \in \operatorname{cl} N_\varepsilon(Q)} \|\mathrm{id}_{T_xM}\| = 1. \tag{3.7}$$

Let $\lambda(t,x,u) := \lambda_{\max}(S\nabla\tilde{F}_{u(t)}(\tilde{\varphi}_{t,u}(x)))$ for all $(t,x,u) \in \mathbb{R}_0^+ \times M \times \mathcal{U}$. Then, by the Wazewski Inequality (Theorem 1.2.18), we obtain

$$\begin{aligned}
L_\varepsilon(\tau) &\leq \sup_{(t,x,u) \in \mathcal{D}(\tau)} \exp\left(\int_0^t \lambda(s,x,u)ds\right) \\
&\leq \sup_{(t,x,u) \in \mathcal{D}(\tau)} \exp\left(\int_0^t \max\{0, \lambda(s,x,u)\}ds\right) \\
&\leq \sup_{(x,u) \in \operatorname{cl} N_\varepsilon(Q) \times \mathcal{U}} \exp\left(\int_0^\tau \max\{0, \lambda(s,x,u)\}ds\right) \\
&\leq \sup_{(x,u) \in \operatorname{cl} N_\varepsilon(Q) \times \mathcal{U}} \exp\left(\tau \operatorname*{ess\,sup}_{t \in [0,\tau]} \max\{0, \lambda(t,x,u)\}\right) \\
&= \sup_{(x,u) \in \operatorname{cl} N_\varepsilon(Q) \times \mathcal{U}} \exp\left(\tau \operatorname*{ess\,sup}_{t \in [0,\tau]} \max\{0, \lambda_{\max}(S\nabla\tilde{F}_{u(t)}(\tilde{\varphi}_{t,u}(x)))\}\right) \\
&\leq \sup_{(z,v) \in \tilde{\varphi}(\mathcal{D}(\tau)) \times U} \exp\left(\tau \max\{0, \lambda_{\max}(S\nabla\tilde{F}_v(z))\}\right).
\end{aligned}$$

By definition of $\tilde{F}$ every solution of system (3.4) starting in $\operatorname{cl} N_\varepsilon(Q)$ stays in $\operatorname{cl} N_{2\varepsilon}(Q)$ for all positive times. Hence, $\tilde{\varphi}(\mathcal{D}(\tau)) \subset \operatorname{cl} N_{2\varepsilon}(Q)$, which by continuity of $(z,v)\mapsto \lambda_{\max}(S\nabla\tilde{F}_v(z))$ implies

$$\begin{aligned} L_\varepsilon(\tau) &\leq \sup_{(z,v)\in \mathrm{cl}\, N_{2\varepsilon}(Q)\times U} \exp\left(\tau \max\{0,\lambda_{\max}(S\nabla\tilde{F}_v(z))\}\right) \\ &= \exp\left(\tau\max\left\{0, \max_{(z,v)\in \mathrm{cl}\, N_{2\varepsilon}(Q)\times U} \lambda_{\max}(S\nabla\tilde{F}_v(z))\right\}\right) < \infty. \end{aligned}$$

Hence, $L_\varepsilon(\tau)\in[1,\infty)$ for all $\tau>0$. We further obtain

$$\frac{1}{\tau}\ln L_\varepsilon(\tau) \leq \sup_{(z,v)\in\tilde{\varphi}(\mathcal{D}(\tau))\times U} \max\left\{0,\lambda_{\max}(S\nabla\tilde{F}_v(z))\right\}. \tag{3.8}$$

<u>*Step 2:*</u> We show that the following estimate holds:

$$h^+_{\mathrm{inv}}(\varepsilon,K,Q) \leq \ln(L_\varepsilon)\dim_F(K). \tag{3.9}$$

To this end, first assume that $L_\varepsilon > 1$. Let $T>0$ be chosen arbitrarily and let $\mathcal{S}^+ = \{(y_1,u_1),\ldots,(y_n,u_n)\}$ be a minimal strongly (T,ε)-spanning set for (K,Q) with respect to system (1.7). (Note that this implies $n = r^+_{\mathrm{inv}}(T,\varepsilon,K,Q)$.) Then, by (3.5), $\mathcal{S}^+$ is also minimal strongly (T,ε)-spanning for (K,Q) with respect to system (3.4). We define

$$K_j := \left\{x\in M \ : \ \max_{t\in[0,T]} d(\tilde{\varphi}(t,x,u_j),\tilde{\varphi}(t,y_j,u_j)) < \varepsilon\right\}, \quad j=1,\ldots,n.$$

By the definition of strongly (T,ε)-spanning sets we have $K\subset\bigcup_{j=1}^n K_j$. Let

$$r(\varepsilon,T) := \varepsilon L_\varepsilon^{-(\lfloor T\rfloor+1)},$$

We want to prove that

$$B_{r(\varepsilon,T)}(y_j)\subset K_j \ \text{ for } j=1,\ldots,n.$$

To this end, let $x\in B_{r(\varepsilon,T)}(y_j)$ be chosen arbitrarily for some $j\in\{1,\ldots,n\}$, and let $t\in[0,T]$ and $s:=t-\lfloor t\rfloor$. By the cocycle property (1.13) $\tilde{\varphi}_{t,u_j}$ decomposes into $\lfloor t\rfloor+1$ maps in the following way:

$$\tilde{\varphi}_{t,u_j} = \tilde{\varphi}_{s,\Theta_{\lfloor t\rfloor}u_j}\circ\tilde{\varphi}_{1,\Theta_{\lfloor t\rfloor-1}u_j}\circ\cdots\circ\tilde{\varphi}_{1,\Theta_1 u_j}\circ\tilde{\varphi}_{1,u_j}.$$

Let $c:[0,1]\to M$ be a shortest geodesic joining x and y_j, which exists by the choice of ε. Since $\tilde{\varphi}_{1,u_j}\circ c$ joins $\tilde{\varphi}(1,x,u_j)$ and $\tilde{\varphi}(1,y_j,u_j)$, we get

$$\begin{aligned} d(\tilde{\varphi}(1,x,u_j),\tilde{\varphi}(1,y_j,u_j)) &\leq \int_0^1 \left\|\tfrac{d}{dr}\tilde{\varphi}_{1,u_j}(c(r))\right\| dr \\ &= \int_0^1 \left\|D\tilde{\varphi}_{1,u_j}(c(r))\dot{c}(r)\right\| dr \\ &\leq \int_0^1 \left\|D\tilde{\varphi}_{1,u_j}(c(r))\right\| \|\dot{c}(r)\| dr \\ &\leq \sup_{(z,v)\in\mathrm{cl}\,N_\varepsilon(Q)\times\mathcal{U}} \|D\tilde{\varphi}_{1,v}(z)\| \int_0^1 \|\dot{c}(r)\|\, dr \\ &\leq L_\varepsilon d(x,y_j) < L_\varepsilon r(\varepsilon,T) = \varepsilon L_\varepsilon^{-\lfloor T\rfloor} \leq \varepsilon. \end{aligned}$$

In the last inequality we used that $L_\varepsilon \geq 1$. Now (if $t \geq 2$) we can choose a shortest geodesic joining $\tilde{\varphi}(1, x, u_j)$ and $\tilde{\varphi}(1, y_j, u_j)$ and estimate the distance of $\tilde{\varphi}(2, x, u_j)$ and $\tilde{\varphi}(2, y_j, u_j)$ in the same way. Recursively, for $l = 1, \ldots, \lfloor t \rfloor - 1$ we obtain

$$\begin{aligned} d(\tilde{\varphi}_{1,\Theta_l u_j} \circ \cdots \circ \tilde{\varphi}_{1,u_j}(x), \tilde{\varphi}_{1,\Theta_l u_j} \circ \cdots \circ \tilde{\varphi}_{1,u_j}(y_j)) &\leq L_\varepsilon^l d(x, y_j) \\ &< \varepsilon L_\varepsilon^{-\lfloor T \rfloor - 1 + l} \leq \varepsilon, \end{aligned}$$

and thus also $d(\tilde{\varphi}_{t,u_j}(x), \tilde{\varphi}_{t,u_j}(y_j)) < \varepsilon$. This proves that $B_{r(\varepsilon,T)}(y_j) \subset K_j$. Now assume to the contrary that $N := N(r(\varepsilon, T), K) < r_{\mathrm{inv}}^+(T, \varepsilon, K, Q) = n$, where $N(r(\varepsilon, T), K)$ denotes the minimal number of $r(\varepsilon, T)$-balls necessary to cover the set K (see Section A.2 of the appendix). Then K can be covered by N balls of radius $r(\varepsilon, T)$, which can be assumed to be centered at points $z_1, \ldots, z_N \in Q$ by Lemma A.2.2. Now we assign to each z_j a control function $v_j \in \mathcal{U}$ such that $(z_j, v_j) \in \mathcal{Q}$, and we define $\tilde{\mathcal{S}}^+ := \{(z_1, v_1), \ldots, (z_N, v_N)\}$. Then $\tilde{\mathcal{S}}^+$ is strongly (T, ε)-spanning for (K, Q), since for every $x \in K$ we have $x \in B_{r(\varepsilon,T)}(z_j)$ for some $j \in \{1, \ldots, N\}$ and we have shown that $d(x, z_j) < r(\varepsilon, T)$ implies $\max_{t \in [0,T]} d(\tilde{\varphi}(t, x, v_j), \tilde{\varphi}(t, z_j, v_j)) < \varepsilon$. Since $\mathcal{S}^+$ is minimal, this is a contradiction. Hence,

$$r_{\mathrm{inv}}^+(T, \varepsilon, K, Q) \leq N(r(\varepsilon, T), K). \tag{3.10}$$

We have $\ln r(\varepsilon, T) = \ln(\varepsilon L_\varepsilon^{-(\lfloor T \rfloor + 1)}) = \ln(\varepsilon) - (\lfloor T \rfloor + 1) \ln(L_\varepsilon)$ and thus

$$T \geq \lfloor T \rfloor = \frac{\ln(\varepsilon) - \ln(r(\varepsilon, T))}{\ln L_\varepsilon} - 1 = -\frac{\ln r(\varepsilon, T)}{\ln L_\varepsilon} \left(1 + \frac{\ln(L_\varepsilon) - \ln(\varepsilon)}{\ln r(\varepsilon, T)}\right). \tag{3.11}$$

Note that $\left(1 + \frac{\ln(L_\varepsilon) - \ln(\varepsilon)}{\ln r(\varepsilon, T)}\right) \to 1$ for $T \to \infty$. This yields

$$\begin{aligned} h_{\mathrm{inv}}^+(\varepsilon, K, Q) &= \limsup_{T \to \infty} \frac{\ln r_{\mathrm{inv}}^+(T, \varepsilon, K, Q)}{T} \\ &\overset{(3.10)}{\leq} \limsup_{T \to \infty} \frac{\ln N(r(\varepsilon, T), K)}{T} \\ &= \ln(L_\varepsilon) \limsup_{T \to \infty} \frac{\ln N(r(\varepsilon, T), K)}{\ln(L_\varepsilon) T} \\ &\overset{(3.11)}{\leq} \ln(L_\varepsilon) \limsup_{T \to \infty} \frac{\ln N(r(\varepsilon, T), K)}{-\ln r(\varepsilon, T) \left(1 + \frac{\ln(L_\varepsilon) - \ln(\varepsilon)}{\ln r(\varepsilon, T)}\right)} \\ &= \ln(L_\varepsilon) \limsup_{T \to \infty} \frac{\ln N(r(\varepsilon, T), K)}{\ln r(\varepsilon, T)^{-1}} \leq \ln(L_\varepsilon) \dim_F(K). \end{aligned}$$

If $L_\varepsilon = 1$, we can prove the same estimate with $L_\varepsilon + \delta = 1 + \delta$ for every $\delta > 0$ and hence, for $\delta \searrow 0$, we obtain $h_{\mathrm{inv}}^+(\varepsilon, K, Q) = 0$.

Step 3: We complete the proof. To this end, consider for every $\tau > 0$ the system

$$\dot{x}(t) = \tau \cdot \tilde{F}(x(t), u(t)), \quad u \in \mathcal{U}. \tag{3.12}$$

Then, by Proposition 2.2.2, Q is also controlled invariant with respect to each of these systems. We denote the corresponding invariance entropy and strong invariance entropy by $h_{\text{inv}}(\varepsilon, K, Q; \tau\tilde{F})$ and $h^+_{\text{inv}}(\varepsilon, K, Q; \tau\tilde{F})$, respectively. By Proposition 2.2.2 we obtain for every $\tau > 0$ the estimate

$$h_{\text{inv}}(\varepsilon, K, Q; \tilde{F}) = \tfrac{1}{\tau} h_{\text{inv}}(\varepsilon, K, Q; \tau\tilde{F}) \overset{(3.2)}{\leq} \tfrac{1}{\tau} h^+_{\text{inv}}(\varepsilon, K, Q; \tau\tilde{F}). \tag{3.13}$$

Now we apply the estimate (3.9) to system (3.12). Denote the cocycle of system (3.12) by $\tilde{\varphi}^\tau$. Then, by the proof of Proposition 2.2.2, we have

$$\tilde{\varphi}^\tau(\tfrac{t}{\tau}, x, \tilde{u}) = \tilde{\varphi}(t, x, u) \quad \text{for all } (t, x, u) \in \mathbb{R} \times M \times \mathcal{U},$$

where $\tilde{u}(t) \equiv u(t\tau)$. Hence,

$$\begin{aligned} \sup_{(t,x,u)\in\mathcal{D}(1)} \left\|D\tilde{\varphi}^\tau_{t,u}(x)\right\| &= \sup_{(t,x,u)\in\mathcal{D}(1)} \|D\tilde{\varphi}_{t\tau,u}(x)\| \\ &= \sup_{(t,x,u)\in\mathcal{D}(\tau)} \|D\tilde{\varphi}_{t,u}(x)\| = L_\varepsilon(\tau). \end{aligned}$$

Consequently, from (3.13) we obtain

$$\begin{aligned} h_{\text{inv}}(\varepsilon, K, Q) &\leq \tfrac{1}{\tau} \ln(L_\varepsilon(\tau)) \dim_F(K) \\ &\overset{(3.8)}{\leq} \sup_{(z,v)\in\tilde{\varphi}(\mathcal{D}(\tau))\times U} \max\{0, \lambda_{\max}(S\nabla\tilde{F}_v(z))\} \dim_F(K) \\ &= \max\left\{0, \sup_{(z,v)\in\tilde{\varphi}(\mathcal{D}(\tau))\times U} \lambda_{\max}(S\nabla\tilde{F}_v(z))\right\} \dim_F(K). \end{aligned}$$

Let $z \in \tilde{\varphi}(D(\tau))$. Then $z = \tilde{\varphi}(t, x, u)$ for some $(t, x, u) \in [0, \tau] \times \text{cl}\, N_\varepsilon(Q) \times \mathcal{U}$. If u is a piecewise constant control function, then the corresponding solution $\tilde{\varphi}(\cdot, x, u)$ is piecewise continuously differentiable, and hence we can measure its length by taking the integral over $\|\frac{d}{dt}\tilde{\varphi}_{x,u}(t)\|$. This implies that for $t \in [0, \tau]$

$$\begin{aligned} d(x, \tilde{\varphi}(t, x, u)) &\leq \int_0^t \left\|\tfrac{d}{dt}\tilde{\varphi}_{x,u}(t)\right\| dt \leq \int_0^\tau \left\|\tfrac{d}{dt}\tilde{\varphi}_{x,u}(t)\right\| dt \\ &= \int_0^\tau \left\|\tilde{F}(\tilde{\varphi}(t, x, u), u(t))\right\| dt \\ &\leq \underbrace{\max_{(z,v)\in\text{cl}\, N_{2\varepsilon}(Q)\times U} \left\|\tilde{F}(z, v)\right\|}_{=:C} \int_0^\tau dt = C\tau. \end{aligned}$$

The same inequality for arbitrary admissible control functions follows from Corollary 1.2.30. This implies

$$\tilde{\varphi}(\mathcal{D}(\tau)) \subset \text{cl}\, N_{\min\{2\varepsilon, \varepsilon+\tau C\}}(Q) \quad \text{for every } \tau > 0.$$

For $\tau > 0$ with $\varepsilon + \tau C < 2\varepsilon$ we obtain

$$h_{\text{inv}}(\varepsilon, K, Q) \leq \max\left\{0, \max_{(z,v)\in\text{cl}\, N_{\varepsilon+\tau C}(Q)\times U} \lambda_{\max}(S\nabla\tilde{F}_v(z))\right\} \dim_F(K).$$

Now take a sequence $(\tau_n)_{n\in\mathbb{N}}$, $\tau_n > 0$, with $\tau_n \searrow 0$. Let $(z_n, v_n) \in \mathrm{cl}\, N_{\varepsilon+\tau_n C}(Q)\times U$ be a point where the maximum above is attained. By compactness we may assume that $(z_n, v_n) \to (z^*, v^*) \in \mathrm{cl}\, N_\varepsilon(Q) \times U$ for $n \to \infty$. Then

$$\lambda_{\max}(S\nabla\tilde{F}_{v^*}(z^*)) = \max_{(z,v)\in \mathrm{cl}\, N_\varepsilon(Q)\times U} \lambda_{\max}(S\nabla\tilde{F}_v(z)), \tag{3.14}$$

since otherwise there exists $(z^{**}, v^{**}) \in \mathrm{cl}\, N_\varepsilon(Q) \times U$ with

$$\lambda_{\max}(S\nabla\tilde{F}_{v^{**}}(z^{**})) > \lambda_{\max}(S\nabla\tilde{F}_{v^*}(z^*)),$$

which, by continuity of $(z, v) \mapsto \lambda_{\max}(S\nabla\tilde{F}_v(z))$, implies

$$\begin{aligned} \lambda_{\max}(S\nabla\tilde{F}_{v_n}(z_n)) &= \max_{(z,v)\in \mathrm{cl}\, N_{\varepsilon+\tau_n C}(Q)\times U} \lambda_{\max}(S\nabla\tilde{F}_v(z)) \\ &< \lambda_{\max}(S\nabla\tilde{F}_{v^{**}}(z^{**})) \\ &\leq \max_{(z,v)\in \mathrm{cl}\, N_\varepsilon(Q)\times U} \lambda_{\max}(S\nabla\tilde{F}_v(z)) \end{aligned}$$

for n large enough. This is a contradiction, since the maximum on $\mathrm{cl}\, N_\varepsilon(Q)\times U$ cannot be greater than the maximum on $\mathrm{cl}\, N_{\varepsilon+\tau_n C}(Q)\times U$. Hence, we conclude that

$$\begin{aligned} h_{\mathrm{inv}}(\varepsilon, K, Q) &\leq \lim_{n\to\infty} \max\left\{0, \max_{(z,v)\in \mathrm{cl}\, N_{\varepsilon+\tau_n C}(Q)\times U} \lambda_{\max}(S\nabla\tilde{F}_v(z))\right\} \dim_F(K) \\ &= \lim_{n\to\infty} \max\left\{0, \lambda_{\max}(S\nabla\tilde{F}_{v_n}(z_n))\right\} \dim_F(K) \\ &= \max\left\{0, \lambda_{\max}(S\nabla\tilde{F}_{v^*}(z^*))\right\} \dim_F(K) \\ &\overset{(3.14)}{=} \max\left\{0, \max_{(z,v)\in \mathrm{cl}\, N_\varepsilon(Q)\times U} \lambda_{\max}(S\nabla\tilde{F}_v(z))\right\} \dim_F(K) \\ &= \max\left\{0, \max_{(z,v)\in \mathrm{cl}\, N_\varepsilon(Q)\times U} \lambda_{\max}(S\nabla F_v(z))\right\} \dim_F(K). \end{aligned}$$

The last equality follows from the fact that $\tilde{F}$ and F coincide on $\mathrm{cl}\, N_\varepsilon(Q) \times U$. With the same arguments it follows that

$$h_{\mathrm{inv}}(K, Q) = \lim_{\varepsilon\searrow 0} h_{\mathrm{inv}}(\varepsilon, K, Q) \leq \max\left\{0, \max_{(z,v)\in Q\times U} \lambda_{\max}(S\nabla F_v(z))\right\} \dim_F(K),$$

which finishes the proof. □

3.1.5 Remark:

In Boichenko & Leonov & Reitmann [8, Corollary 6.2.1, p. 292], one finds a similar estimate for the topological entropy of a flow on $\mathbb{R}^n$, restricted to a compact flow-invariant set. There logarithmic matrix norms are used in order to compute asymptotic Lipschitz constants for the time-one-map of the flow, and then an upper estimate for the topological entropy proved by Ito [30] is applied. In Noack [44, Satz 2.1.1, p. 75], the analogous result is proved for flows on Riemannian manifolds by applying an upper estimate for the topological entropy of a map (involving an asymptotic Lipschitz constant and the lower box dimension of the state space), to the time-t-maps of the flow.

The idea of the following corollary comes from Noack [44, Folgerung 2.11.1, p. 75].

3.1.6 Corollary:
Under the assumptions of Theorem 3.1.4, let $W \subset M$ be an open neighborhood of Q and $\alpha : W \to \mathbb{R}$ a C^∞-function. Then

$$\boxed{h_{\text{inv}}(K,Q) \leq \max\left\{0, \max_{(x,u)\in Q\times U}\left(\lambda_{\max}(S\nabla F_u(x)) + \mathcal{L}_{F_u}\alpha(x)\right)\right\} \cdot \dim_F(K).}$$

Proof:
We define a new Riemannian metric $\tilde{g}$ on W by

$$\tilde{g}(x) := e^{2\alpha(x)}g(x) \quad \text{for all } x \in W$$

and we let $\tilde{\nabla}$ denote the Levi-Civita connection associated with $\tilde{g}$. Then, by Lemma A.3.6, for every $f \in \mathcal{X}^1(M)$ the matrix representation of $S\tilde{\nabla}f$ with respect to a chart (ϕ, V) is given by

$$\begin{aligned}
2\left[S\tilde{\nabla}f\right]_{\mu\nu} &= \partial_\nu\phi(f^\mu) + \sum_{\theta,\kappa}\partial_\theta\phi(f^\kappa)\tilde{g}^{\mu\theta}\tilde{g}_{\kappa\nu} + \sum_{i,l} f^i\tilde{g}^{\mu l}\frac{\partial\tilde{g}_{\nu l}}{\partial x_i} \\
&= \partial_\nu\phi(f^\mu) + \sum_{\theta,\kappa}\partial_\theta\phi(f^\kappa)g^{\mu\theta}g_{\kappa\nu} + \sum_{i,l} f^i e^{-2\alpha}g^{\mu l}\frac{\partial(e^{2\alpha}g_{\nu l})}{\partial x_i} \\
&= \partial_\nu\phi(f^\mu) + \sum_{\theta,\kappa}\partial_\theta\phi(f^\kappa)g^{\mu\theta}g_{\kappa\nu} \\
&\quad + e^{-2\alpha}\sum_{i,l} f^i g^{\mu l}\left[e^{2\alpha}\frac{\partial g_{\nu l}}{\partial x_i} + 2e^{2\alpha}g_{\nu l}\frac{\partial\alpha}{\partial x_i}\right] \\
&= \partial_\nu\phi(f^\mu) + \sum_{\theta,\kappa}\partial_\theta\phi(f^\kappa)g^{\mu\theta}g_{\kappa\nu} \\
&\quad + \sum_{i,l} f^i g^{\mu l}\frac{\partial g_{\nu l}}{\partial x_i} + 2\sum_{i,l} f^i g^{\mu l}g_{l\nu}\frac{\partial\alpha}{\partial x_i}.
\end{aligned}$$

Since $\sum_l g^{\mu l}g_{l\nu} = \delta_{\mu\nu}$, we obtain

$$\left[S\tilde{\nabla}f\right]_{\mu\nu} = [S\nabla f]_{\mu\nu} + \delta_{\mu\nu}\sum_i f^i\frac{\partial\alpha}{\partial x_i} = [S\nabla f]_{\mu\nu} + (\mathcal{L}_f\alpha)\delta_{\mu\nu}.$$

This implies the assertion. □

3.1.7 Example:
Consider the one-dimensional linear control system

$$\dot{x}(t) = ax(t) + u(t), \quad u \in \mathcal{U}, \tag{3.15}$$

where $a \in \mathbb{R}$ and $U = [c,d]$ for some $c,d \in \mathbb{R}$ with $c \leq d$. With respect to the standard metric on $\mathbb{R}$ the symmetrized covariant derivative of the right-hand side $F(x,u) = ax + u$ equals a constantly. Hence, Theorem 3.1.4 implies

$$\boxed{h_{\text{inv}}(K,Q) \leq \max\{0,a\} \cdot \dim_F(K).} \tag{3.16}$$

In the next section, we will show that equality holds in (3.16). $\Diamond$

3.1.8 Example:
Consider the bilinear control system (2.18) from Example 2.2.13 and its projection to the unit sphere. On S^{d-1} consider the round metric and let $Q \subset S^{d-1}$ be a compact controlled invariant set, and $K \subset Q$ compact. We want to use the estimate (3.3) to determine an upper bound for $h_{\text{inv}}(K, Q)$. To this end, we must compute the symmetrized covariant derivative of the right-hand side of the projected system. By Example 2.2.13, this right-hand side is given by

$$F(s,u) = (A(u) - s^T A(u) s I)s, \quad S^{d-1} \times \mathbb{R}^m \to TS^{d-1}.$$

Let us consider F as a mapping from $\mathbb{R}^d \times \mathbb{R}^m$ to $\mathbb{R}^d$ for a moment and compute its derivative with respect to s:

$$D_1 F(s,u) = A(u) - s^T A(u) s I - s s^T (A(u) + A(u)^T). \tag{3.17}$$

Then, by Gallot & Hulin & Lafontaine [22, Proposition 2.56, p. 69], the covariant derivative $\nabla_v F_u(s)$ is given by the orthogonal projection of $D_1 F(s,u)v$ onto $T_s S^{d-1} = s^\perp$. Writing $Q_s := I - ss^T$ and using that $(I - ss^T)ss^T = 0$, we obtain

$$\nabla_v F(s) = Q_s (A(u) - s^T A(u) s I) v. \tag{3.18}$$

The adjoint operator $\nabla F(s)^* : T_s S^{d-1} \to T_s S^{d-1}$ is the unique linear mapping on $T_s S^{d-1}$ with the property $g_s(\nabla_v F(s), w) = g_s(v, \nabla_w F(s)^*)$ for all $v, w \in T_s S^{d-1}$. Since g_s is just the restriction of the Euclidean scalar product of $\mathbb{R}^d$ to $T_s S^{d-1}$, we obtain

$$\begin{aligned} g_s\left(\nabla_v F(s), w\right) &= \left\langle Q_s (A(u) - s^T A(u) s I) v, w \right\rangle \\ &= \left\langle v, (A(u)^T - s^T A(u) s I) Q_s w \right\rangle \\ &= \left\langle Q_s v, (A(u)^T - s^T A(u) s I) w \right\rangle \\ &= \left\langle v, Q_s (A(u)^T - s^T A(u) s I) w \right\rangle. \end{aligned}$$

Hence, $\nabla_v F(s)^* = Q_s (A(u)^T - s^T A(u) s I) v$, which implies

$$\begin{aligned} S\nabla_v F_u(s) &= \tfrac{1}{2}\left[\nabla_v F(s) + \nabla_v F(s)^*\right] \\ &= \tfrac{1}{2} Q_s \left[A(u) + A(u)^T - 2 s^T A(u) s I\right] v. \end{aligned}$$

Writing $A(u)^+$ for $\frac{1}{2}(A(u) + A(u)^T)$ and using that $s^T A(u) s = s^T A(u)^+ s$, we obtain

$$S\nabla F_u(s) = Q_s A(u)^+ - s^T A(u)^+ s I.$$

Hence, Theorem 3.1.4 implies

$$\boxed{h_{\text{inv}}(K,Q) \leq \max\left\{0, \max_{(s,u) \in Q \times U} \lambda_{\max}\left(Q_s A(u)^+ - (s^T A(u)^+ s) I\right)\right\} \cdot \dim_F(K).}$$

In the next section, we will also determine a lower bound for projected bilinear systems on the sphere. $\Diamond$

3.1.9 Open Question:
Is it possible to find a Riemannian metric such that the estimate (3.3) becomes optimal?

3.2 Lower Bounds

The first main theorem of this section yields a lower bound for the invariance entropy in terms of the divergence of the right-hand side with respect to a volume form on M. In the proof, we use a volume growth argument to determine lower bounds for the numbers $r_{\text{inv}}(T,\varepsilon,K,Q)$. In order to apply this argument we have to assume that the set K has positive volume.

3.2.1 Theorem (General Lower Bound):
Consider control system (1.7). Let $Q \subset M$ be a compact controlled invariant set and $K \subset Q$ compact. Let ω be a volume form on M of class C^1 and assume that $\mu_\omega(K) > 0$. Then the following estimate holds:

$$\boxed{h_{\text{inv}}(K,Q) \geq \max\left\{0, \min_{(x,u)\in Q\times U} \operatorname{div}_\omega F_u(x)\right\}.} \tag{3.19}$$

Proof:
For arbitrary $T,\varepsilon > 0$ let $\mathcal{S} = \{u_1,\ldots,u_n\}$ be a minimal (T,ε)-spanning set for (K,Q) and define

$$K_j := \{x \in K \mid \varphi([0,T],x,u_j) \subset N_\varepsilon(Q)\}, \quad j = 1,\ldots,n.$$

Then, by definition of (T,ε)-spanning sets, $K = \bigcup_{j=1}^n K_j$. For each $j \in \{1,\ldots,n\}$ the set K_j is a Borel set, since it is the intersection of the compact set K and the open set $\{x \in M \mid \varphi([0,T],x,u_j) \subset N_\varepsilon(Q)\}$. By Corollary 1.2.12 φ_{T,u_j} is a homeomorphism and therefore also $\varphi_{T,u_j}(K_j)$ is a Borel set. Hence, we get

$$\mu_\omega(\varphi_{T,u_j}(K_j)) \leq \mu_\omega(N_\varepsilon(Q)), \quad j = 1,\ldots,n. \tag{3.20}$$

For the ω-measure of $\varphi_{T,u_j}(K_j)$ we obtain

$$\begin{aligned}
\mu_\omega(\varphi_{T,u_j}(K_j)) &= \int_{\varphi_{T,u_j}(K_j)} d\mu_\omega \stackrel{(A.20)}{=} \int_{K_j} |\det_\omega D\varphi_{T,u_j}(x)| d\mu_\omega(x) \\
&\geq \int_{K_j} d\mu_\omega \inf_{\substack{(x,u)\in K\times\mathcal{U} \\ \varphi([0,T],x,u)\subset N_\varepsilon(Q)}} |\det_\omega D\varphi_{T,u}(x)| \\
&= \mu_\omega(K_j) \inf_{\substack{(x,u)\in K\times\mathcal{U} \\ \varphi([0,T],x,u)\subset N_\varepsilon(Q)}} |\det_\omega D\varphi_{T,u}(x)|.
\end{aligned}$$

By the Liouville Formula (Theorem 1.2.16) this implies

$$\mu_\omega(\varphi_{T,u_j}(K_j)) \geq \mu_\omega(K_j) \cdot \inf_{\substack{(x,u)\in K\times\mathcal{U} \\ \varphi([0,T],x,u)\subset N_\varepsilon(Q)}} \exp\left(\int_0^T \operatorname{div}_\omega F_{u(s)}(\varphi(s,x,u))ds\right).$$

Let

$$V(\varepsilon,T) := \inf_{\substack{(x,u)\in K\times\mathcal{U} \\ \varphi([0,T],x,u)\subset N_\varepsilon(Q)}} \exp\left(\int_0^T \operatorname{div}_\omega F_{u(s)}(\varphi(s,x,u))ds\right).$$

By Lemma A.3.2 we may assume that ε is chosen small enough that $\operatorname{cl} N_\varepsilon(Q)$ is compact. For every $(x,u) \in K \times \mathcal{U}$ with $\varphi([0,T],x,u) \subset N_\varepsilon(Q)$ it holds that

$$\begin{aligned}\exp\left(\int_0^T \operatorname{div}_\omega F_{u(s)}(\varphi(s,x,u))ds\right) &\geq \exp\left(T \min_{(z,u)\in \operatorname{cl} N_\varepsilon(Q)\times U} \operatorname{div}_\omega F_u(z)\right) \\ &= \min_{(z,u)\in \operatorname{cl} N_\varepsilon(Q)\times U} \exp\left(T \operatorname{div}_\omega F_u(z)\right),\end{aligned}$$

which implies

$$V(\varepsilon,T) \geq \min_{(z,u)\in \operatorname{cl} N_\varepsilon(Q)\times U} \exp\left(T \operatorname{div}_\omega F_u(z)\right) > 0. \tag{3.21}$$

We obtain

$$\mu_\omega(K_j) \leq \frac{\mu_\omega(\varphi_{T,u_j}(K_j))}{V(\varepsilon,T)} \overset{(3.20)}{\leq} \frac{\mu_\omega(N_\varepsilon(Q))}{V(\varepsilon,T)}. \tag{3.22}$$

Let $j_0 \in \{1,\dots,n\}$ be chosen such that $\mu_\omega(K_{j_0}) = \max_{j=1,\dots,n} \mu_\omega(K_j)$. Then

$$\mu_\omega(K) \leq \mu_\omega\Big(\bigcup_{j=1}^n K_j\Big) \leq n \cdot \mu_\omega(K_{j_0}) \overset{(3.22)}{\leq} n \cdot \frac{\mu_\omega(N_\varepsilon(Q))}{V(\varepsilon,T)}.$$

Since $n = r_{\mathrm{inv}}(T,\varepsilon,K,Q)$, we get

$$r_{\mathrm{inv}}(T,\varepsilon,K,Q) \geq \frac{\mu_\omega(K)}{\mu_\omega(N_\varepsilon(Q))} V(\varepsilon,T) \quad \text{for all } T,\varepsilon > 0$$

and hence

$$\begin{aligned} h_{\mathrm{inv}}(\varepsilon,K,Q) &\geq \limsup_{T\to\infty}\Big[\frac{1}{T}\ln V(\varepsilon,T) + \underbrace{\frac{1}{T}\ln\frac{\mu_\omega(K)}{\mu_\omega(N_\varepsilon(Q))}}_{\to 0}\Big] \\ &\overset{(3.21)}{\geq} \limsup_{T\to\infty} \min_{(z,u)\in \operatorname{cl} N_\varepsilon(Q)\times U} \operatorname{div}_\omega F_u(z) \\ &= \min_{(x,u)\in \operatorname{cl} N_\varepsilon(Q)\times U} \operatorname{div}_\omega F_u(x). \end{aligned}$$

For $\varepsilon \searrow 0$ we have $\min_{(x,u)\in \operatorname{cl} N_\varepsilon(Q)\times U} \operatorname{div}_\omega F_u(x) \to \min_{(x,u)\in Q\times U} \operatorname{div}_\omega F_u(x)$, which can be seen as follows: Assume to the contrary that there exists $\delta > 0$ such that for all $n \in \mathbb{N}$ there is $(x_n,u_n) \in \operatorname{cl} N_{1/n}(Q) \times U$ with

$$\operatorname{div}_\omega F_{u_n}(x_n) = \min_{(x,u)\in \operatorname{cl} N_{1/n}(Q)\times U} \operatorname{div}_\omega F_u(x)$$

and

$$\min_{(x,u)\in Q\times U} F_u(x) - \operatorname{div}_\omega F_{u_n}(x_n) \geq \delta.$$

By compactness of $\operatorname{cl} N_{1/n}(Q) \times U$ we may assume that (x_n,u_n) converges to some $(x_*,u_*) \in Q \times U$, which, by continuity of $(x,u) \mapsto \operatorname{div}_\omega F_u(x)$, leads to the contradiction

$$\operatorname{div}_\omega F_{u_*}(x_*) + \delta \leq \min_{(x,u)\in Q\times U} F_u(x) \leq \operatorname{div}_\omega F_{u_*}(x_*).$$

Hence, the assertion is true. □

3.2.2 Remark:
In Gelfert [23] one finds lower estimates for the topological entropy of dynamical systems in terms of so-called *singular value functions* and *global Lyapunov exponents*. These estimates are also based on volume growth arguments, but the proof techniques are quite different than ours, which essentially is due to the fact that the topological entropy—roughly speaking—measures the maximal speed at which solutions move away from each other, which is determined by the local (or even the infinitesimal) behavior of the system, while the invariance entropy depends on the global behavior of the system on the controlled invariant set Q, which may not be determined by local properties. (For example, if Q is the whole state space, the invariance entropy is always zero, no matter how complicated the dynamics of the system are.) Further estimates for the entropy of a dynamical system involving volume growth rates are due to Newhouse [43] and Yomdin [54]. They derived upper bounds for the metric entropy of a diffeomorphism on a compact manifold in terms of the volume growth rate of embedded submanifolds under the iterated application of the diffeomorphism.

3.2.3 Example:
Consider again the bilinear control (2.18) and its projection to the unit sphere:

$$\dot{s}(t) = F(s(t), u(t)), \quad u \in \mathcal{U}, \quad F(s,u) = \left[A(u) - s^T A(u) s I\right] s.$$

Consider the volume form on S^{d-1} induced by the round metric, and let $Q \subset S^{d-1}$ be a compact controlled invariant set, and $K \subset Q$ a compact set of positive measure. We want to use estimate (3.19) to determine a lower bound for $h_{\mathrm{inv}}(K,Q)$. To this end, we must compute the divergence of F_u, which is given by the trace of the covariant derivative (see Formula (A.22)). Let $(v_1, \ldots, v_{d-1})$ be an orthonormal basis of $T_s S^{d-1}$. Then, with (3.18) we get

$$\begin{aligned} \operatorname{div} F_u(s) &= \operatorname{tr} \nabla F_u(s) = \sum_{i=1}^{d-1} \langle Q_s(A(u) - s^T A(u) s I) v_i, v_i \rangle \\ &= \sum_{i=1}^{d-1} \langle (A(u) - s^T A(u) s I) v_i, Q_s v_i \rangle \\ &= \sum_{i=1}^{d-1} \langle (A(u) - s^T A(u) s I) v_i, v_i \rangle \\ &= \operatorname{tr}(A(u) - s^T A(u) s I) - \underbrace{\langle (A(u) - s^T A(u) s I) s, s \rangle}_{=0} \\ &= \operatorname{tr} A(u) - d \cdot s^T A(u) s. \end{aligned}$$

Hence, the estimate

$$\boxed{h_{\mathrm{inv}}(K,Q) \geq \max \left\{ 0, \min_{(s,u) \in Q \times U} \left(\operatorname{tr} A(u) - d \cdot s^T A(u) s \right) \right\}}$$

follows from Theorem 3.2.1. $\Diamond$

3.2.4 Corollary:
Under the assumptions of Theorem 3.2.1 let $\alpha : W \to \mathbb{R}$ be a C^1-function, defined on an open neighborhood W of Q. Then

$$\boxed{h_{\mathrm{inv}}(K,Q) \geq \max\left\{0, \min_{(x,u)\in Q\times U}\left[\mathrm{div}_\omega F_u(x) + \mathcal{L}_{F_u}\alpha(x)\right]\right\}.} \tag{3.23}$$

Proof:
On W consider the volume form $\omega' := \beta\cdot\omega$ with $\beta(x) \equiv e^{\alpha(x)}$. Using a cut-off function we can extend ω' to M. Then by Formula (A.18) we have

$$\begin{aligned}\mathrm{div}_{\omega'} F_u(x) &= \mathrm{div}_\omega F_u(x) + \frac{\mathcal{L}_{F_u}\beta(x)}{\beta(x)}\\ &= \mathrm{div}_\omega F_u(x) + \frac{e^{\alpha(x)}\mathcal{L}_{F_u}\alpha(x)}{e^{\alpha(x)}} = \mathrm{div}_\omega F_u(x) + \mathcal{L}_{F_u}\alpha(x).\end{aligned}$$

Now the assertion immediately follows from Theorem 3.2.1. □

3.2.5 Example:
Let $M = (0,\infty)$ and $F(x,u) = 2ax + 2\sqrt{x}u$ with $a > 0$, $F : M\times\mathbb{R} \to \mathbb{R}$. Consider the control system

$$\dot{x}(t) = F(x(t), u(t)), \quad u \in \mathcal{U},$$

with control range $U = [-1, -\frac{1}{2}]$. Let $Q := [\frac{1}{4a^2}, \frac{1}{a^2}]$. Then Q is controlled invariant, since for every $x \in Q$ one can define the constant control function $u_x(t) :\equiv -a\sqrt{x} \in [-1,-\frac{1}{2}]$, which yields $F(x,u_x) = 0$. Using the standard volume form one obtains the estimate

$$h_{\mathrm{inv}}(Q) \geq \min_{(x,u)\in Q\times U}\frac{\partial F}{\partial x}(x,u) = \min_{(x,u)\in Q\times U}\left[2a + \frac{u}{\sqrt{x}}\right] = 0,$$

since $\frac{-1}{(4a^2)^{-1/2}} = -2a$. Now let $\alpha : (0,\infty) \to \mathbb{R}$ be given by $\alpha(x) := -\frac{1}{2}\ln(x)$. Then, by Corollary 3.2.4, we obtain

$$\begin{aligned}h_{\mathrm{inv}}(K,Q) &\geq \min_{(x,u)\in Q\times U}\left[\mathrm{div}\, F_u(x) + \alpha'(x)F_u(x)\right]\\ &= \min_{(x,u)\in Q\times U}\left[2a + \frac{u}{\sqrt{x}} - a - \frac{u}{\sqrt{x}}\right] = a.\end{aligned}$$

This shows that the standard volume form is not always the best choice for estimating the invariance entropy from below. ◊

3.2.6 Corollary:
Consider a control-affine system of the form

$$\dot{x}(t) = f_0(x(t)) + u(t)f_1(x(t)), \quad u \in \mathcal{U},$$

on a smooth oriented d-dimensional manifold M endowed with a Riemannian metric g of class C^∞, and $f_0, f_1 \in \mathcal{X}^\infty(M)$. Let $Q \subset M$ be compact and controlled invariant and let $K \subset Q$ be a compact set with positive Riemannian volume. Moreover, assume that there exists an open neighborhood W of Q and a vector field $h \in \mathcal{X}^\infty(W)$ such that the following hypotheses are satisfied:

(i) $f_1(x) \neq 0$ *for all* $x \in Q$.

(ii) $g_x(f_1(x), h(x)) = 0$ *for all* $x \in Q$.

(iii) *The following vector field is integrable on* W:[2]

$$x \mapsto h(x) - \frac{\operatorname{div}_g f_1(x)}{\|f_1(x)\|_x^2} f_1(x).$$

Then $h_{\text{inv}}(K, Q)$ *is bounded from below by*

$$\boxed{\max\left\{0, \min_{x \in Q}\left[\operatorname{div}_g f_0(x) - \tfrac{\operatorname{div}_g f_1(x)}{\|f_1(x)\|_x^2} g_x(f_0(x), f_1(x)) + g_x(f_0(x), h(x))\right]\right\},}$$

where div_g *denotes the divergence with respect to the volume form induced by the Riemannian metric* g *(cf. Formula* (A.22)*).*

Proof:
Let $F(x, u) := F_u(x) := f_0(x) + uf_1(x)$. By hypothesis (iii) there exists a C^∞-function $\alpha : W \to \mathbb{R}$ such that

$$\operatorname{grad} \alpha(x) = h(x) - \frac{\operatorname{div}_g f_1(x)}{\|f_1(x)\|_x^2} f_1(x) \quad \text{for all } x \in W. \tag{3.24}$$

On W we define a new Riemannian metric by

$$\tilde{g}(x) := e^{\frac{2}{d}\alpha(x)} g(x) \quad \text{for all } x \in W.$$

Let $\tilde{\nabla}$ denote the Levi-Civita connection associated with $\tilde{g}$. Then from Formula (A.21) and the proof of Corollary 3.2.4 it easily follows that for any vector field $f \in \mathcal{X}^\infty(W)$ one has

$$\operatorname{div}_{\tilde{g}} f(x) = \operatorname{tr} \tilde{\nabla} f(x) = \operatorname{tr} \nabla f(x) + (\mathcal{L}_f \alpha)(x) = \operatorname{div}_g f(x) + (\mathcal{L}_f \alpha)(x).$$

From the definition of the gradient (see (A.15)) it follows that

$$\mathcal{L}_f(\alpha)(x) = f(\alpha)(x) = g_x(\operatorname{grad} \alpha(x), f(x)) \tag{3.25}$$

for every vector field $f \in \mathcal{X}^\infty(W)$. Hence, for all $(x, u) \in Q \times U$ we obtain

$$\begin{aligned}
\operatorname{div}_{\tilde{g}} F_u(x) \quad &= \quad \operatorname{div}_g F_u(x) + (\mathcal{L}_{F_u} \alpha)(x) = \operatorname{div}_g f_0(x) + u \operatorname{div}_g f_1(x) \\
&+ \quad f_0(\alpha)(x) + uf_1(\alpha)(x) \overset{(3.25)}{=} \operatorname{div}_g f_0(x) + u \operatorname{div}_g f_1(x) \\
&+ \quad g_x(f_0(x), \operatorname{grad} \alpha(x)) + ug_x(f_1(x), \operatorname{grad} \alpha(x)) \\
&\overset{(3.24)}{=} \quad \operatorname{div}_g f_0(x) + u \operatorname{div}_g f_1(x) + g_x(f_0(x), h(x)) \\
&- \quad \frac{\operatorname{div}_g f_1(x)}{\|f_1(x)\|_x^2} g_x(f_0(x), f_1(x)) + u \underbrace{g_x(f_1(x), h(x))}_{=0} \\
&- \quad u \frac{\operatorname{div}_g f_1(x)}{\|f_1(x)\|_x^2} \underbrace{g_x(f_1(x), f_1(x))}_{=\|f_1(x)\|_x^2} \\
&= \quad \operatorname{div}_g f_0(x) + g_x(f_0(x), h(x)) - \frac{\operatorname{div}_g f_1(x)}{\|f_1(x)\|_x^2} g_x(f_0(x), f_1(x)).
\end{aligned}$$

[2] This means that the vector field is the gradient of some C^∞-function.

Thus, the result follows from Theorem 3.2.1. □

Next, we define a property of control systems which guarantees that the invariance entropy of a set with positive fractal dimension is positive.

3.2.7 Definition (Uniformly Expanding System):
Consider control system (1.7) and let $Q \subset M$ be compact. Assume that there exist $\varepsilon, c, \lambda > 0$ such that for all $x_1, x_2 \in Q$, $u \in \mathcal{U}$ and $T > 0$ with $\varphi([0,T], x_j, u) \subset N_\varepsilon(Q)$ $(j = 1, 2)$ the estimate

$$d(\varphi(T, x_1, u), \varphi(T, x_2, u)) \geq ce^{\lambda T} d(x_1, x_2) \tag{3.26}$$

holds. Then the system is called **uniformly expanding** *on Q with respect to the metric d. The constant λ is called an* **expansion factor**.

The following proposition provides a simple condition for the right-hand side of a system which guarantees that the system is uniformly expanding.

3.2.8 Proposition:
Consider control system (1.7). Let g be a complete Riemannian metric on M of class C^∞. Let $\varepsilon, \rho > 0$ be real numbers such that

$$\lambda_{\min}(S\nabla F_u(x)) \geq \rho \quad \text{for all } (x, u) \in \mathrm{gh}(N_\varepsilon(Q)) \times U, \tag{3.27}$$

where $\lambda_{\min}(\cdot)$ denotes the minimal eigenvalue and $\mathrm{gh}(N_\varepsilon(Q))$ the union of the images of all shortest geodesics joining points in $N_\varepsilon(Q)$.[3] *Then the system is uniformly expanding on Q with expansion factor ρ.*

Proof:
We subdivide the proof into three steps. First we prove expansiveness for constant control functions, then for piecewise constant ones and finally, for arbitrary admissible control functions.

Step 1: Let $x_1, x_2 \in Q$, $T > 0$ and $u \in \mathcal{U}$ a constant control function, say $u(t) \equiv u_0 \in U$. Assume further that $\varphi([0,T], x_j, u) \subset N_\varepsilon(Q)$ for $j = 1, 2$. In order to prove expansiveness with expansion factor ρ we show the following:

$$\forall \delta \in (0, \rho): \; d(\varphi(T, x_1, u), \varphi(T, x_2, u)) \geq e^{\delta T} d(x_1, x_2). \tag{3.28}$$

To this end, we consider the time-reversed system

$$\dot{x}(t) = f(x(t)), \quad f(x) :\equiv -F(x, u_0).$$

The time-t-map of the corresponding flow is denoted by $\phi^t : M \to M$. It obviously holds that $\phi^t \equiv \varphi_{-t,u_0}$. Hence, (3.28) is equivalent to

$$\forall \delta \in (0, \rho): \; d(x_1, x_2) \leq e^{-\delta T} d(\phi^{-T}(x_1), \phi^{-T}(x_2)). \tag{3.29}$$

[3] The letters "gh" are supposed to stand for "geodesic hull", though we mean something slightly different here.

With the substitution $\tilde{x}_j := \phi^{-T}(x_j) = \varphi(T, x_j, u)$, $j = 1, 2$, we obtain the equivalent statement

$$\forall \delta \in (0, \rho) : \; d(\phi^T(\tilde{x}_1), \phi^T(\tilde{x}_2)) \leq e^{-\delta T} d(\tilde{x}_1, \tilde{x}_2). \tag{3.30}$$

In order to prove (3.30) we introduce for every $\tau > 0$ the set

$$A(\tau) := \bigcup_{t \in [0,\tau]} \phi^t(\mathrm{gh}(N_\varepsilon(Q)))$$

and show that the following statement holds:

$$\forall \delta \in (0, \rho) : \exists \tau > 0 : \; \forall (x, u) \in A(\tau) \times U : \; \lambda_{\min}(S \nabla F_u(x)) \geq \delta. \tag{3.31}$$

To this end, first note that $\mathrm{gh}(N_\varepsilon(Q))$ is relatively compact, which follows from the assumption that (M, g) is complete and $\mathrm{gh}(N_\varepsilon(Q))$ is bounded, since obviously $\mathrm{diam}\, \mathrm{gh}(N_\varepsilon(Q)) = \mathrm{diam}(N_\varepsilon(Q))$. For every $\delta \in (0, \rho)$ we find a neighborhood W of $\mathrm{cl}\, \mathrm{gh}(N_\varepsilon(Q))$ such that $\lambda_{\min}(S \nabla F_u(X)) \geq \delta$ holds for all $(x, u) \in W \times U$, which follows the fact that the map $(x, u) \mapsto \lambda_{\min}(S \nabla F_u(x))$ is uniformly continuous on the compact set $\mathrm{cl}\, \mathrm{gh}(N_\varepsilon(Q)) \times U$. Hence, it suffices to show that τ can be chosen small enough such that $A(\tau) \subset W$. Assume to the contrary that there is no such τ. Then we find sequences $(t_n)_{n \in \mathbb{N}}$ in $(0, \infty)$ with $t_n \to 0$ and $(x_n)_{n \in \mathbb{N}}$ in $\mathrm{gh}(N_\varepsilon(Q))$ converging to some point $x \in \mathrm{cl}\, \mathrm{gh}(N_\varepsilon(Q))$ such that $\phi^{t_n}(x_n) \in M \backslash W$ for all $n \in \mathbb{N}$. By continuity the contradiction $x = \phi^0(x) \in (M \backslash W) \cap \mathrm{cl}\, \mathrm{gh}(N_\varepsilon(Q)) = \emptyset$ follows. Hence, (3.31) is proved. Now let $t \in [0, \tau]$ for some $\tau = \tau(\delta)$. Let $c : [0, 1] \to M$ be a shortest geodesic from $\tilde{x}_1$ to $\tilde{x}_2$, which exists by completeness of (M, g). Then we have

$$\begin{aligned}
d(\phi^t(\tilde{x}_1), \phi^t(\tilde{x}_2)) \; &\leq \mathcal{L}(\phi^t \circ c) = \int_0^1 \left\| \tfrac{d}{ds}(\phi^t \circ c)(s) \right\| ds \\
&= \int_0^1 \| D\phi^t(c(s)) \dot{c}(s) \| ds \leq \int_0^1 \| D\phi^t(c(s)) \| \cdot \| \dot{c}(s) \| ds \\
&\leq \left(\max_{\xi \in c([0,1])} \| D\phi^t(\xi) \| \right) \int_0^1 \| \dot{c}(s) \| ds \\
&\leq \left(\max_{\xi \in \mathrm{cl}\, \mathrm{gh}(N_\varepsilon(Q))} \| D\phi^t(\xi) \| \right) d(\tilde{x}_1, \tilde{x}_2).
\end{aligned}$$

By the Wazewski Inequality (Theorem 1.2.18) we obtain the estimate

$$\| D\phi^t(\xi) \| \leq \exp\left(t \sup_{s \in [0,t]} \lambda(s) \right),$$

where $\lambda(s)$ denotes the maximal eigenvalue of $S \nabla f(\phi^s(\xi)) = -S \nabla F_u(\phi^s(\xi))$. Since $\phi^s(\xi) \in A(\tau)$ for all $s \in [0, \tau]$, (3.31) implies

$$\begin{aligned}
\| D\phi^t(\xi) \| \; &\leq \exp\left(t \sup_{s \in [0,t]} [-\lambda_{\min}(S \nabla F_u(\phi^s(\xi)))] \right) \\
&\leq \exp\left(t \sup_{s \in [0,t]} (-\delta) \right) = \exp(-\delta t).
\end{aligned}$$

Hence, for all $t \in [0, \tau]$ we have

$$d(\phi^t(\tilde{x}_1), \phi^t(\tilde{x}_2)) \leq e^{-\delta t} d(\tilde{x}_1, \tilde{x}_2).$$

An inductive argument (using the flow property) shows that the same estimate holds for arbitrary $T > 0$. Hence, we have proved (3.30).

Step 2: Let $x_1, x_2 \in Q$, $u \in \mathcal{U}$ a control function and $T > 0$ any number such that $\varphi([0,T], x_j, u) \subset N_\varepsilon(Q)$, $j = 1, 2$, and such that there exists a partition $0 = t_0 < t_1 < \cdots < t_n = T$ with $u(t)$ constant on each of the intervals $[t_{j-1}, t_j]$, $j = 1, \ldots, n$. Then, by Step 1, we have

$$d(\varphi_{t,u}(x_1), \varphi_{t,u}(x_2)) \geq e^{t\rho} d(x_1, x_2) \quad \text{for all } t \in [t_0, t_1].$$

Now let $t \in [t_1, t_2]$. Then the cocycle property (1.13) implies

$$d(\varphi_{t,u}(x_1), \varphi_{t,u}(x_2)) = d(\varphi(t - t_1, \varphi(t_1, x_1, u), \Theta_{t_1} u), \varphi(t - t_1, \varphi(t_1, x_2, u), \Theta_{t_1} u)).$$

Since $\Theta_{t_1} u$ is constant on $[0, t_2 - t_1]$, again Step 1 implies

$$\begin{aligned} d(\varphi_{t,u}(x_1), \varphi_{t,u}(x_2)) &\geq e^{(t - t_1)\rho} d(\varphi_{t_1,u}(x_1), \varphi_{t_1,u}(x_2)) \\ &\geq e^{(t-t_1)\rho} e^{t_1 \rho} d(x_1, x_2) = e^{t\rho} d(x_1, x_2). \end{aligned}$$

Inductively we obtain the assertion for all $t \in [0, T]$.

Step 3: Expansiveness for arbitrary admissible control functions now can easily be concluded using Corollary 1.2.30. □

3.2.9 Theorem (Lower Bound for Expanding Systems):
Assume that the system (1.7) is uniformly expanding on Q with expansion factor λ. Let $Q \subset M$ be compact and controlled invariant, and let $K \subset Q$ be compact. Then the following estimate holds:

$$\boxed{h_{\mathrm{inv}}(K, Q) \geq \lambda \cdot \dim_F(K).} \tag{3.32}$$

Proof:
Let $\mathcal{S} = \{u_1, \ldots, u_n\}$ be a minimal (T, ε)-spanning set for (K, Q), where ε is chosen small enough according to Definition 3.2.7 and T is arbitrary. Define

$$K_j := \{x \in K \mid \varphi([0,T], x, u_j) \subset N_\varepsilon(Q)\}, \quad j = 1, \ldots, n.$$

Then $\{K_1, \ldots, K_n\}$ is a covering of K and by minimality $K_j \neq \emptyset$ for $j = 1, \ldots, n$. Let $x, y \in K_j$ for some $j \in \{1, \ldots, n\}$. Then by (3.26) it follows that

$$c e^{\lambda T} d(x, y) \leq d(\varphi(T, x, u_j), \varphi(T, y, u_j)) \leq \operatorname{diam} N_\varepsilon(Q),$$

which implies

$$d(x, y) \leq \frac{\operatorname{diam} N_\varepsilon(Q)}{c} e^{-\lambda T}.$$

Let $C(\varepsilon) := \frac{\operatorname{diam} N_\varepsilon(Q)}{c}$. Then K_j is contained in the ball with radius $C(\varepsilon) e^{-\lambda T}$ centered at any point in K_j and hence

$$r_{\mathrm{inv}}(T, \varepsilon, K, Q) = n \geq N(C(\varepsilon) e^{-\lambda T}, K). \tag{3.33}$$

It holds that $\ln([C(\varepsilon)e^{-\lambda T}]^{-1}) = \lambda T - \ln C(\varepsilon)$, and thus

$$T = \frac{\ln(C(\varepsilon)^{-1}e^{\lambda T}) + \ln C(\varepsilon)}{\lambda} = \frac{\ln(C(\varepsilon)^{-1}e^{\lambda T})}{\lambda}\left(1 + \frac{\ln C(\varepsilon)}{\ln(C(\varepsilon)^{-1}e^{\lambda T})}\right). \quad (3.34)$$

Note that

$$\lim_{T\to\infty}\left(1 + \frac{\ln C(\varepsilon)}{\ln(C(\varepsilon)^{-1}e^{\lambda T})}\right) = 1. \quad (3.35)$$

Hence, we obtain

$$\begin{aligned}
h_{\mathrm{inv}}(\varepsilon, K, Q) &= \limsup_{T\to\infty}\frac{1}{T}\ln r_{\mathrm{inv}}(T,\varepsilon,K,Q) \\
&\overset{(3.33)}{\geq} \limsup_{T\to\infty}\frac{1}{T}\ln N(C(\varepsilon)e^{-\lambda T}, K) \\
&= \lambda\limsup_{T\to\infty}\frac{\ln N(C(\varepsilon)e^{-\lambda T}, K)}{\lambda T} \\
&\overset{(3.34)}{=} \lambda\limsup_{T\to\infty}\frac{\ln N(C(\varepsilon)e^{-\lambda T}, K)}{\lambda\frac{\ln(C(\varepsilon)^{-1}e^{\lambda T})}{\lambda}\left(1 + \frac{\ln C(\varepsilon)}{\ln(C(\varepsilon)^{-1}e^{\lambda T})}\right)} \\
&= \lambda\limsup_{T\to\infty}\frac{\ln N(C(\varepsilon)e^{-\lambda T}, K)}{\ln(C(\varepsilon)^{-1}e^{\lambda T})\left(1 + \frac{\ln C(\varepsilon)}{\ln(C(\varepsilon)^{-1}e^{\lambda T}}\right)} \\
&\overset{(3.35)}{=} \lambda\limsup_{T\to\infty}\frac{\ln N(C(\varepsilon)e^{-\lambda T}, K)}{\ln(C(\varepsilon)^{-1}e^{\lambda T})} = \lambda\dim_F(K).
\end{aligned}$$

For $\varepsilon \searrow 0$ the assertion follows. □

3.2.10 Example:
Consider the one-dimensional linear system (3.15) from Example 3.1.7. By Proposition 1.1.12 the solutions are given by

$$\varphi(t,x,u) \equiv e^{at}x + \int_0^t e^{a(t-s)}u(s)ds.$$

This implies that for all $x, y \in \mathbb{R}$, $u \in \mathcal{U}$ and $t \geq 0$ it holds that

$$|\varphi(t,x,u) - \varphi(t,y,u)| = e^{at}|x-y|.$$

Hence, for $a > 0$, the system is uniformly expanding on every compact set with expansion factor a and thus Theorem 3.2.9 implies $h_{\mathrm{inv}}(K,Q) \geq a\cdot\dim_F(K)$. Together with (3.16) we obtain

$$\boxed{h_{\mathrm{inv}}(K,Q) = \max\{0,a\}\cdot\dim_F(K)} \quad (3.36)$$

for arbitrary $a \in \mathbb{R}$. ◊

3.2.11 Remark:
From Formula (3.36) it follows that in general $h_{\mathrm{inv}}(K,Q) \neq \sup_{j\in\mathbb{N}} h_{\mathrm{inv}}(K_j,Q)$ if $K = \bigcup_{j\in\mathbb{N}} K_j$. As a counterexample, consider the linear system (3.15) with

$a = 1$ and control range $U = [-1, 1]$. Then $Q := [-1, 1]$ is controlled invariant, since every point $x \in Q$ becomes an equilibrium for the constant control function $u_x(t) :\equiv -x$. The set $K := \{0\} \cup \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ is compact, and by Boichenko & Leonov & Reitmann [8, Example 2.2.2, p. 198] its fractal dimension is $\frac{1}{2}$. Now let $K_0 := \{0\}$ and $K_j := \{\frac{1}{j}\}$ for every $j \in \mathbb{N}$. Then $K = \bigcup_{j \in \mathbb{N}_0} K_j$, but

$$h_{\mathrm{inv}}(K, Q) \stackrel{(3.36)}{=} \frac{1}{2} \neq 0 = \sup_{j \in \mathbb{N}_0} \underbrace{h_{\mathrm{inv}}(K_j, Q)}_{=0}.$$

3.2.12 Open Question:

Is it possible to find a volume form such that the estimate (3.19) becomes optimal?

Chapter 4

Relation to Lyapunov Exponents

This chapter deals with the invariance entropy of linear control systems and of control sets for control-affine systems. In both cases we will see that there is a strong connection between the invariance entropy and the Lyapunov exponents of the system, i.e., the exponential growth rates

$$\lambda(x,u;z) = \limsup_{t\to\infty} \frac{1}{t} \ln \|D\varphi_{t,u}(x)z\|$$

for $(x,u) \in M \times \mathcal{U}$ and $z \in T_xM$, where the norm $\|\cdot\|$ is induced by some Riemannian metric. For a linear control system $\dot{x} = Ax + Bv$, $v \in \mathcal{V}$, the invariance entropy $h_{\text{inv}}(K,Q)$ is given by the sum of the eigenvalues of A with positive real parts (counted with their multiplicities), provided that the set K has positive Lebesgue measure. The proof of this result is partially based on Bowen's theorem on the topological entropy of a linear map [10, Theorem 15]. For the more general inhomogeneous bilinear systems, where the constant matrix A is replaced by a control-dependent matrix of the form $A(u) = A_0 + \sum_i u_i A_i$, we still obtain that $h_{\text{inv}}(K,Q)$ is bounded from below by the sum of the positive minimal Lyapunov exponents on subbundles of $\mathcal{U} \times \mathbb{R}^d$, which are invariant under the control flow of the bilinear system $\dot{x} = A(u)x$, $u \in \mathcal{U}$. Here again we need the assumption that K has positive Lebesgue measure. Unfortunately, we cannot provide an analogous upper bound for systems of this type.

For control-affine systems the controlled invariance of a control set carries over to its closure. Hence, if the closure of a control set D is compact, the (strict) invariance entropy $h_{\text{inv}}^{(*)}(K, \operatorname{cl} D)$ is defined. From the approximate controllability on D it can be concluded that $h_{\text{inv}}^{(*)}(K, \operatorname{cl} D)$ does not depend on the set K, provided that K is contained in D and has nonvoid interior. Both for linear systems and for one-dimensional systems with one control vector field it can be shown that $h_{\text{inv}}(K, \operatorname{cl} D) = h_{\text{inv}}^*(K, \operatorname{cl} D)$. In the one-dimensional case it turns out that, under mild conditions, the entropy is the maximum of zero and the minimal Lyapunov exponent of the system restricted to $\operatorname{cl} D$. As an example, the invariance entropies of the control sets of a controlled linear oscillator are computed. Finally, the following theorem is proved: If the linearization along a controlled periodic trajectory in the interior of a control set D is controllable, then $h^*_{\text{inv}}(K, \operatorname{cl} D)$ is bounded from above by the sum of the positive Lyapunov exponents of the periodic solution. The proof of this theorem is a modification of the proof of Theorem 3 in Nair & Evans & Mareels & Moran [42], which says that the local topological feedback entropy of a discrete-time control system at a fixed point is given by the sum of the unstable eigenvalues of the fixed point Jacobian. As an example, we compute the upper bound of the theorem explicitly for an equilibrium of a projected bilinear system on the unit sphere.

Since the Lyapunov exponents describe the stability behavior of the solutions, where the positive Lyapunov exponents correspond to unstable solutions, the results of this chapter suggest that, in general, the invariance entropy of a set Q is the higher the more unstable the solutions in Q are. For the metric entropy and the topological entropy of differentiable dynamical systems similar relations to the Lyapunov exponents are known. For example, the metric entropy of a C^1-diffeomorphism on a compact manifold with respect to an invariant Borel measure μ is bounded from above by the μ-integral over the sum of the positive Lyapunov exponents on each tangent space. This estimate, known as the *Ruelle Inequality*, can be found, e.g., in Katok & Hasselblatt [33, Theorem S.2.13, p. 669]. For a C^2-diffeomorphism with a smooth invariant measure the Ruelle Inequality becomes an equality (see Pesin [45]). In Gelfert [23] lower bounds of the topological entropy in terms of Lyapunov exponents are derived both for maps and for flows of class C^1 (see [23, Folgerung 6.1.1 and Folgerung 6.1.2, pp. 112–113]).

4.1 Linear and Inhomogeneous Bilinear Systems

Throughout this section, we consider control-affine systems on $\mathbb{R}^d$ of the form

$$\dot{x}(t) = \left[A_0 + \sum_{i=1}^{m_1} u_i(t) A_i\right] x(t) + Bv(t), \quad (u,v) \in \mathcal{U} \times \mathcal{V}, \tag{4.1}$$

where $A_0, A_1, \ldots, A_{m_1} \in \mathbb{R}^{d\times d}$ and $B \in \mathbb{R}^{d\times m_2}$ ($m = m_1 + m_2$), and the control range is a product $U \times V$ of compact sets $U \subset \mathbb{R}^{m_1}$ and $V \subset \mathbb{R}^{m_2}$. We call a system of this type an *inhomogeneous bilinear control system*. In the case that $m_1 = 0$ the system is called a *linear control system* and takes the form

$$\dot{x}(t) = Ax(t) + Bv(t), \quad v \in \mathcal{V}. \tag{4.2}$$

We frequently use the abbreviation

$$A(u) := A_0 + \sum_{i=1}^{m_1} u_i A_i.$$

By Proposition 1.1.12 the solution of (4.1) is given by

$$\varphi(t,x,w) = \varphi(t,x,(u,v)) = \Lambda_u(t,0)x + \int_0^t \Lambda_u(t,s)Bv(s)ds, \tag{4.3}$$

where $\Lambda_u(t,s)$ is the evolution operator corresponding to the linear homogeneous equation $\dot{x}(t) = A(u(t))x(t)$.

4.1.1 Notation:
In the proofs of this section, we frequently work with the internal and the external sum of vector spaces. In order to avoid confusion, we use the following notation: For linear subspaces W_1 and W_2 of a real vector space V we write

$$W_1 \oplus_i W_2 = \{w_1 + w_2 \mid w_1 \in W_1 \text{ and } w_2 \in W_2\}$$

for the internal sum of W_1 and W_2, and

$$W_2 \oplus_o W_2 = \{(w_1, w_2) \mid w_1 \in W_1 \text{ and } w_2 \in W_2\} = W_1 \times W_2$$

for the external sum of W_1 and W_2.

Our first result yields a formula for the invariance entropy of the linear system (4.2) under the assumption that the set K has positive Lebesgue measure.

4.1.2 Theorem (Invariance Entropy of Linear Systems):
Consider the linear control system (4.2). Let $K, Q \subset \mathbb{R}^d$ be compact sets with $K \subset Q$ and Q being controlled invariant. Let $\lambda_1, \ldots, \lambda_d$ be the eigenvalues of A. Then the following estimate holds:

$$\boxed{h^+_{\text{inv}}(K,Q) \leq \sum_{i:\ \text{Re}(\lambda_i)>0} \text{Re}(\lambda_i).}$$

If, in addition, K has positive Lebesgue measure, we have

$$\boxed{h_{\text{inv}}(K,Q) = h^+_{\text{inv}}(K,Q) = \sum_{i:\ \text{Re}(\lambda_i)>0} \text{Re}(\lambda_i).}$$

Proof:
The proof is subdivided into three steps.

<u>*Step 1:*</u> We show that $h^+_{\text{inv}}(K,Q)$ is bounded from above by the sum of the positive real parts of the eigenvalues of A. To this end, consider the linear semiflow $\Phi(t,x) = e^{At}x$, $\Phi : \mathbb{R}^+_0 \times \mathbb{R}^d \to \mathbb{R}^d$. With respect to the Euclidean norm, this semiflow is uniformly continuous in the sense of Definition A.2.5, since for all $t_0 > 0$, $t \in [0,t_0]$ and $x, y \in \mathbb{R}^d$ one has

$$\|e^{At}x - e^{At}y\| = \|e^{At}(x-y)\| \leq \|e^{At}\|\|x-y\| \leq \left(\max_{t\in[0,t_0]} \|e^{At}\|\right) \|x-y\|.$$

Hence, by Proposition A.2.6 the topological entropy $h_{\text{top}}(\Phi)$ equals the topological entropy of the time-one-map $\Phi_1(x) = e^{A}x$. By Proposition A.2.7(i) the topological entropy of the linear map Φ_1 is given by

$$h_{\text{top}}(\Phi_1) = \sum_{i:\ |\mu_i|>1} \ln|\mu_i|,$$

where $\mu_1, \ldots, \mu_d$ are the eigenvalues of e^A. Since $|\mu_i| = \left|e^{\lambda_i}\right| = e^{\mathrm{Re}(\lambda_i)}$, we obtain

$$h_{\mathrm{top}}(\Phi) = h_{\mathrm{top}}(\Phi_1) = \sum_{i:\ |e^{\lambda_i}|>1} \mathrm{Re}(\lambda_i) = \sum_{i:\ \mathrm{Re}(\lambda_i)>0} \mathrm{Re}(\lambda_i).$$

Hence, it suffices to show that $h^+_{\mathrm{inv}}(K,Q) \leq h_{\mathrm{top}}(\Phi)$. To this end, for given $T, \varepsilon > 0$ let $E \subset Q$ be a maximal (T,ε)-separated set with respect to the semiflow Φ, say $E = \{y_1, \ldots, y_n\}$. Then E is also (T,ε)-spanning the set Q, which means that for all $x \in Q$ there is $j \in \{1, \ldots, n\}$ with

$$\max_{t\in[0,T]} \|e^{At}x - e^{At}y_j\| < \varepsilon.$$

Since Q is controlled invariant, we can assign to each y_j ($j \in \{1, \ldots, n\}$) a control function $v_j \in \mathcal{V}$ such that $\varphi(\mathbb{R}^+_0, y_j, v_j) \subset Q$. Let $\mathcal{S}^+ := \{(y_1, v_1), \ldots, (y_n, v_n)\} \subset \mathcal{Q}$. Since $\varphi(t,x,v) - \varphi(t,y,v) = e^{At}x - e^{At}y$ for all $t \geq 0$, $x, y \in \mathbb{R}^d$ and $v \in \mathcal{V}$, we obtain that $\mathcal{S}^+$ is strongly (T,ε)-spanning for (Q,Q) and hence also for (K,Q). This implies

$$r^+_{\mathrm{inv}}(T,\varepsilon,K,Q) \leq s(T,\varepsilon,Q,\Phi) \quad \text{for all } T,\varepsilon > 0$$

and consequently, $h^+_{\mathrm{inv}}(K,Q) \leq h_{\mathrm{sep}}(Q,\Phi) = h_{\mathrm{top}}(Q,\Phi) \leq h_{\mathrm{top}}(\Phi)$.

<u>*Step 2:*</u> Under the assumption that $\lambda^d(K) > 0$ and $\mathrm{Re}(\lambda_i) > 0$ for all $i \in \{1, \ldots, d\}$ we prove that

$$h_{\mathrm{inv}}(K,Q) \geq \sum_{i=1}^{d} \mathrm{Re}(\lambda_i).$$

This is an easy consequence of Theorem 3.2.1: With $F_v(x) = Ax + Bv$ we obtain

$$\operatorname{div}_{\omega_0} F_v(x) = \operatorname{tr} DF_v(x) = \operatorname{tr} A = \sum_{i=1}^{d} \lambda_i = \sum_{i=1}^{d} \mathrm{Re}(\lambda_i),$$

where ω_0 is the standard volume form on $\mathbb{R}^d$. The last equality holds, since nonreal eigenvalues of a real matrix appear as pairs of complex conjugate numbers and thus the imaginary parts in the sum cancel. By (3.19) the assertion follows.

<u>*Step 3:*</u> We prove the inequality $h_{\mathrm{inv}}(K,Q) \geq \sum_{i:\ \mathrm{Re}(\lambda_i)>0} \mathrm{Re}(\lambda_i)$ under the assumption $\lambda^d(K) > 0$ for arbitrary matrices A: If all real parts of the eigenvalues of A are nonpositive, the assertion is true, since $h_{\mathrm{inv}}(K,Q) \geq 0$ holds anyway. Hence, we may assume that there exists at least one eigenvalue with positive real part. We write $\mathbb{E}^s$, $\mathbb{E}^u$ and $\mathbb{E}^c$ for the corresponding stable, unstable and center subspace with respect to the flow $(t,x) \mapsto e^{At}x$. This furnishes the decomposition $\mathbb{R}^d = \mathbb{E}^u \oplus_i (\mathbb{E}^s \oplus_i \mathbb{E}^c)$. Consider the projection

$$\pi : \mathbb{R}^d \to \mathbb{E}^u, \quad x \mapsto x^u.$$

The map π is obviously of class C^1 and we can project the control system (4.2) to $\mathbb{E}^u$: Let $F(x,v) = Ax+Bv$ and $G(y,v) = A\,|_{\mathbb{E}^u}\, y+\pi Bv$, $G : \mathbb{E}^u \times \mathbb{R}^m \to \mathbb{E}^u$. Then we have

$$D\pi_x F(x,v) = \pi(Ax+Bv) = \pi Ax + \pi Bv = A\pi x + \pi Bv = G(\pi x, v)$$

and thus we can apply Proposition 2.2.8, which yields

$$h_{\text{inv}}(K,Q;F) \geq h_{\text{inv}}(\pi K, \pi Q; G).$$

Since the projected system on $\mathbb{E}^u$ also is a linear control system and all real parts of the eigenvalues of $A\,|_{\mathbb{E}^u}\colon \mathbb{E}^u \to \mathbb{E}^u$ are positive, by Step 2 we obtain

$$h_{\text{inv}}(K,Q) \geq \sum_{\lambda\in\sigma(A|\mathbb{E}^u)} \text{Re}(\lambda) = \sum_{i:\ \text{Re}(\lambda_i)>0} \text{Re}(\lambda_i),$$

if $\pi K \subset \mathbb{E}^u$ has positive Lebesgue measure. In order to show the latter, let $s = \dim \mathbb{E}^u$ and let λ^s denote the s-dimensional Lebesgue measure on $\mathbb{E}^u$. Assume to the contrary that $\lambda^s(\pi K) = 0$, and consider the linear transformation

$$\alpha : \mathbb{R}^d \to \text{im}(\pi) \oplus_o \ker(\pi), \quad x \mapsto (\pi x, x - \pi x).$$

On $\text{im}(\pi) \oplus_o \ker(\pi)$ let $\langle\cdot,\cdot\rangle_e$ be the inner product given by

$$\langle (u_1,v_1),(u_2,v_2)\rangle_e = \langle u_1,u_2\rangle + \langle v_1,v_2\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the standard inner product of $\mathbb{R}^d$. The inner product $\langle\cdot,\cdot\rangle_e$ induces a norm $\|\cdot\|_e$ and a Lebesgue measure λ_e^d on $\text{im}(\pi) \oplus_o \ker(\pi)$. Using the transformation theorem and the theorem of Fubini we obtain

$$\begin{aligned}
\lambda^d(K) \;&\leq\; \lambda^d(\pi^{-1}\pi K) = \int_{\mathbb{R}^d} \mathbb{1}_{\pi^{-1}\pi K}(x)dx \\
&= \int_{\text{im}(\pi)\oplus_o\ker(\pi)} \mathbb{1}_{\pi^{-1}\pi K}(\alpha^{-1}(u,v))|\det \alpha^{-1}|d(u,v) \\
&= |\det \alpha^{-1}| \int_{\text{im}(\pi)} \int_{\ker(\pi)} \mathbb{1}_{\pi^{-1}\pi K}(u+v)du\ dv \\
&= |\det \alpha^{-1}| \int_{\text{im}(\pi)} \int_{\ker(\pi)} \mathbb{1}_{\alpha(\pi^{-1}\pi K)}(u,v)du\ dv.
\end{aligned}$$

Since $\alpha(\pi^{-1}\pi K) = \pi K \times \ker(\pi)$, we obtain the contradiction

$$\begin{aligned}
\lambda^d(K) \;&\leq\; |\det \alpha^{-1}| \int_{\text{im}(\pi)} \int_{\ker(\pi)} \mathbb{1}_{\pi K\times\ker(\pi)}(u,v)du\ dv \\
&= |\det \alpha^{-1}| \int_{\text{im}(\pi)} \int_{\ker(\pi)} \mathbb{1}_{\pi K}(u)\mathbb{1}_{\ker(\pi)}(v)dv\ du \\
&= |\det \alpha^{-1}| \int_{\ker(\pi)} \mathbb{1}_{\ker(\pi)}(v) \left(\int_{\text{im}(\pi)} \mathbb{1}_{\pi K}(u)du \right) dv \\
&= |\det \alpha^{-1}| \int_{\ker(\pi)} \mathbb{1}_{\ker(\pi)}(v)\, \underbrace{\lambda^s(\pi K)}_{=0}\, dv = 0.
\end{aligned}$$

This finishes the proof. $\square$

4.1.3 Remarks:

- In the case when $\lambda^d(K) = 0$ we cannot make a general statement about the exact value of $h_{\mathrm{inv}}(K,Q)$ unless $d = 1$ (see Example 3.2.10). If, e.g., K is finite, then $h_{\mathrm{inv}}(K,Q) = 0$. But if the projection of K to $\mathbb{E}^u(A)$ has positive Lebesgue measure in $\mathbb{E}^u(A)$, then $h_{\mathrm{inv}}(K,Q) = \sum_{i:\ \mathrm{Re}(\lambda_i)>0} \mathrm{Re}(\lambda_i)$ anyway.
- The existence of a nonvoid compact controlled invariant subset for the linear control system (4.2) can be guaranteed, if the pair (A,B) is controllable, the matrix A is hyperbolic and the control range U is compact and convex with nonvoid interior. Then there exists a unique control set D with nonvoid interior and its closure $Q = \mathrm{cl}\, D$ is compact (see Colonius & Spadini [17, Theorem 4.1]). It is easily seen to be controlled invariant (see Lemma 2.3.1).

Next, we consider one-dimensional inhomogeneous bilinear systems. For simplicity, we assume that $m_1 = m_2 = 1$:

$$\dot{x}(t) = [a + u(t)]x(t) + v(t), \quad (u(t), v(t)) \in \underbrace{[u_{\min}, u_{\max}]}_{=U} \times \underbrace{[v_{\min}, v_{\max}]}_{=V}. \tag{4.4}$$

The solution of (4.4) is given by

$$\varphi(t,x,(u,v)) = e^{\int_0^t (a+u(s))ds} \left(x + \int_0^t v(s) e^{-\int_0^s (a+u(\tau))d\tau} ds \right), \tag{4.5}$$

as can easily be verified. Using the estimates we already know, we can prove the following proposition.

4.1.4 Proposition:

Consider system (4.4) and let $K, Q \subset \mathbb{R}$ be compact with $K \subset Q$ and Q being controlled invariant. Then the following statements hold:

(i) *If $a + u_{\max} \leq 0$, then $h_{\mathrm{inv}}(K,Q) = 0$.*

(ii) *If $a + u_{\min} \geq 0$, then $h_{\mathrm{inv}}(K,Q) \in [a + u_{\min}, a + u_{\max}] \cdot \dim_F(K)$.*

Proof:

Theorem 3.1.4 yields

$$h_{\mathrm{inv}}(K,Q) \leq \max\{0, a + u_{\max}\} \cdot \dim_F(K), \tag{4.6}$$

which implies (i). From (4.5) it follows that

$$|\varphi(t,x,(u,v)) - \varphi(t,y,(u,v))| = e^{\int_0^t (a+u(s))ds} |x - y|.$$

This implies that for $a + u_{\min} > 0$ the system is uniformly expanding on every compact set with expansion factor $a + u_{\min}$. Theorem 3.2.9 implies that

$$h_{\mathrm{inv}}(K,Q) \geq (a + u_{\min}) \cdot \dim_F(K).$$

Together with (4.6) this proves (ii). □

4.1.5 Remark:
Note that the interval $[a+u_{\min}, a+u_{\max}]$ coincides with the Lyapunov spectrum of system (4.4).

Now consider system (4.1) in arbitrary dimensions. Assume that the sets U and V are convex. Then also $U \times V$ is convex and by Proposition 1.3.14 the cocycle $\varphi : \mathbb{R} \times \mathbb{R}^d \times (\mathcal{U} \times \mathcal{V}) \to \mathbb{R}^d$ is continuous, since the system is control-affine.

In the following, we denote the space of all projections $P \in \mathbb{R}^{d \times d}$ with k-dimensional image by $\mathcal{P}(k, d, \mathbb{R})$. If W is a k-dimensional subspace of $\mathbb{R}^d$, then we have a k-dimensional Lebesgue measure λ_W^k on W, since W can be identified isometrically with $\mathbb{R}^k$. We will use several times that images of certain measurable sets under projections are again measurable with respect to the lower-dimensional Lebesgue measure in the image space of the projection. While in general this is not true,[1] it holds for sets which can be written as the intersection of an open and a compact set:

4.1.6 Lemma:
Let $P \in \mathcal{P}(k, d, \mathbb{R})$ be a projection and $A = K \cap U \subset \mathbb{R}^d$, where K is compact and U is open. Then $P(A) \subset \operatorname{im}(P)$ is Lebesgue measurable with respect to the k-dimensional Lebesgue measure in $\operatorname{im}(P)$.

Proof:
By Cohn [13, Proposition 1.1.5, p. 6] the open set U is an F_σ-set, i.e., the countable union of closed sets C_n, $n \in \mathbb{N}$. Hence,

$$P(A) = P(K \cap U) = P\Big(K \cap \bigcup_{n \in \mathbb{N}} C_n\Big) = P\Big(\bigcup_{n \in \mathbb{N}} (K \cap C_n)\Big) = \bigcup_{n \in \mathbb{N}} P(K \cap C_n).$$

Since C_n is closed and K is compact, $C_n \cap K$ is compact and by continuity of P also $P(K \cap C_n)$. Every compact set is closed and thus $P(A)$ again is an F_σ-set. Since F_σ-sets are Borel sets, $P(A) \subset \operatorname{im}(P)$ is Lebesgue measurable. □

In order to prove our next result we need the following two technical lemmas.

4.1.7 Lemma:
For arbitrary $z \in \mathbb{R}^d$ and $r > 0$ the function

$$\mu : \mathcal{P}(k, d, \mathbb{R}) \to \mathbb{R}_0^+, \quad P \mapsto \lambda_{\operatorname{im}(P)}^k(P(B_r(z))),$$

is bounded on every compact subset of $\mathcal{P}(k, d, \mathbb{R})$.

Proof:
Any projection $P \in \mathcal{P}(k, d, \mathbb{R})$, regarded as a mapping from $\mathbb{R}^d$ to $\operatorname{im}(P)$, is an open map by the open mapping theorem. Consequently, $P(B_r(z)) \subset \operatorname{im}(P)$ is open and thus a Borel set. This shows that μ is well-defined.

[1] Counterexample: Let $P : \mathbb{R}^2 \to \mathbb{R}$, $P(x, y) = x$, and let $A \subset \mathbb{R}$ be a set which is not Lebesgue measurable with respect to the one-dimensional Lebesgue measure. Then $A \times \{0\} \subset \mathbb{R}^2$ is Lebesgue measurable with respect to the two-dimensional Lebesgue measure, since it is contained in a set of measure zero (the x-axis), and $P(A \times \{0\}) = A$.

In order to prove the assertion, consider for fixed $P \in \mathcal{P}(k,d,\mathbb{R})$ the affine subspace $W := z + \ker(P)^\perp$. The restriction $P|_W \colon W \to \operatorname{im}(P)$ is bijective. In order to show this, assume $P(z+x_1) = P(z+x_2)$ for some $x_1, x_2 \in \ker(P)^\perp$. Then $P(x_1 - x_2) = 0$ and consequently, $x_1 - x_2 \in \ker(P) \cap \ker(P)^\perp = \{0\}$. In particular, P maps $\ker(P)^\perp$ bijectively onto $\operatorname{im}(P)$ because of injectivity, linearity and $\dim \ker(P)^\perp = \dim \operatorname{im}(P)$. This yields

$$P(W) = P(z + \ker(P)^\perp) = Pz + P(\ker(P)^\perp) = Pz + \operatorname{im}(P) = \operatorname{im}(P).$$

We want to show that $W \cap B_r(z)$ is mapped onto $P(B_r(z))$. To this end, let $y \in P(B_r(z))$. Then $y = Px$ for some $x \in \mathbb{R}^d$ with $\|x - z\| < r$. We have to prove that there exists $\tilde{x} \in W$, $\tilde{x} = z + \tilde{w}$, with $\|\tilde{x} - z\| < r$ and $P\tilde{x} = Px$. Let $\tilde{x}$ be the orthogonal projection of x onto W, i.e.,

$$\tilde{x} := z + \sum_{j=1}^{k} \langle e_j, x - z\rangle e_j \in z + \ker(P)^\perp = W.$$

where $(e_j)_{j=1}^k$ is an orthonormal basis of $\ker(P)^\perp$. We extend $(e_j)_{j=1}^k$ to an orthonormal basis $(e_j)_{j=1}^d$ of $\mathbb{R}^d$. Then we obtain

$$\begin{aligned} \|\tilde{x} - z\|^2 &= \left\| \sum_{j=1}^{k} \langle e_j, x - z\rangle e_j \right\|^2 = \sum_{j=1}^{k} |\langle e_j, x - z\rangle|^2 \\ &\leq \sum_{j=1}^{d} |\langle e_j, x - z\rangle|^2 = \left\| \sum_{j=1}^{d} \langle e_j, x - z\rangle e_j \right\| = \|x - z\|^2 < r^2. \end{aligned}$$

We have $x = \sum_{j=1}^d \langle x, e_j\rangle e_j$ and $z = \sum_{j=1}^d \langle z, e_j\rangle e_j$. Thus,

$$x - \tilde{x} = \sum_{j=k+1}^{d} \langle x, e_j\rangle e_j - \sum_{j=k+1}^{d} \langle z, e_j\rangle e_j \in \ker(P).$$

This shows that $Px = P\tilde{x}$. Now we know that $P(B_r(z))$ can be parametrized over $D := \ker(P)^\perp \cap B_r(0)$ by the affine map $x \mapsto Pz + Px$. Let

$$L(P) := P|_{\ker(P)^\perp} \colon \ker(P)^\perp \to \operatorname{im}(P).$$

Then the transformation rule yields

$$\mu(P) = \lambda^k_{\operatorname{im}(P)}(P(B_r(z))) = \lambda^k_{\ker(P)^\perp}(D) |\det L(P)|.$$

The k-dimensional Lebesgue measure of D in $\ker(P)^\perp$ does not depend on P because of the symmetry of the ball $B_r(0)$. It is just the volume of a k-dimensional ball of radius r. Hence, it suffices to show that the function $P \mapsto |\det L(P)|$ is bounded on compact sets $\mathcal{P} \subset \mathcal{P}(k,d,\mathbb{R})$. With

$$\left\| P|_{\ker(P)^\perp} \right\| = \max_{\substack{v \in \ker(P)^\perp \\ \|v\| = 1}} \|Pv\|.$$

we obtain

$$\|Px\| \leq \|P|_{\ker(P)^\perp}\| \|x\| \quad \text{for all } x \in \ker(P)^\perp. \tag{4.7}$$

Let $\alpha_1, \ldots, \alpha_k \geq 0$ be the singular values of $L(P)$, i.e., the eigenvalues of the self-adjoint operator $\sqrt{L(P)^* L(P)} : \ker(P)^\perp \to \ker(P)^\perp$. Then there exists an orthonormal basis $(v_1, \ldots, v_k)$ of $\ker(P)^\perp$ of corresponding eigenvectors, i.e., $L(P)^* L(P) v_i = \alpha_i^2 v_i$ and $\langle v_i, v_j \rangle = \delta_{ij}$ for $i, j = 1, \ldots, k$. By (4.7) we obtain

$$\begin{aligned} \|P|_{\ker(P)^\perp}\|^2 &= \|P|_{\ker(P)^\perp}\|^2 \|v_i\|^2 \geq \|Pv_i\|^2 = \|Lv_i\|^2 \\ &= \langle Lv_i, Lv_i \rangle = \langle L^* L v_i, v_i \rangle = \alpha_i^2 \|v_i\|^2 = \alpha_i^2. \end{aligned}$$

Hence,

$$|\det L(P)| = \prod_{i=1}^k \alpha_i \leq \|P|_{\ker(P)^\perp}\|^k \leq \|P\|^k.$$

The assertion now follows by continuity of $P \mapsto \|P\|^k$. □

4.1.8 Lemma:
Let $K \subset \mathbb{R}^d$ be a compact set with positive Lebesgue measure and let $\mathcal{P} \subset \mathcal{P}(k, d, \mathbb{R})$ be a compact set of projections with k-dimensional image. For all $P \in \mathcal{P}(k, d, \mathbb{R})$ let λ_P^k denote the k-dimensional Lebesgue measure in $\operatorname{im}(P)$. Then there exists $\beta > 0$ such that for every finite open covering $\{K_1, \ldots, K_r\}$ of the set K (i.e., $K = \bigcup_{j=1}^r K_j$ and the sets K_j are open relative to K) and for every $P_1, \ldots, P_r \in \mathcal{P}$ we have

$$\sum_{j=1}^r \lambda_{P_j}^k (P_j(K_j)) \geq \beta.$$

Proof:
It suffices to show that there exists a constant $C > 0$ such that

$$\lambda^d(A) \leq C \lambda_P^k(P(A)) \quad \text{for all } P \in \mathcal{P} \text{ and measurable } A \subset K. \tag{4.8}$$

Under the assumption that (4.8) holds we can prove the assertion by contradiction: Assume that β does not exist. Then we can find a sequence of coverings $(\mathcal{K}_n)_{n \in \mathbb{N}}$, $\mathcal{K}_n = \{K_1^n, \ldots, K_{r_n}^n\}$ and a corresponding sequence of tuples $(P_1^n, \ldots, P_{r_n}^n) \in \mathcal{P}^{r_n}$ such that

$$\sum_{j=1}^{r_n} \lambda_{P_j^n}^k (P_j^n(K_j^n)) \to 0 \quad \text{for} \quad n \to \infty.$$

Since $\lambda^d(K) > 0$ by assumption, for sufficiently large $n \in \mathbb{N}$ this leads to the contradiction

$$\lambda^d(K) \leq \sum_{j=1}^{r_n} \lambda^d(K_j^n) \overset{(4.8)}{\leq} C \sum_{j=1}^{r_n} \lambda_{P_j^n}^k (P_j^n(K_j^n)) < \lambda^d(K).$$

In order to prove (4.8) let $\hat{K}$ be a compact ball with $K \subset \hat{K}$. Then we obtain for every $P \in \mathcal{P}$ and for every measurable set $A \subset K$:

$$\lambda^d(A) \le \lambda^d(P^{-1}(P(A)) \cap K) \le \lambda^d(P^{-1}(P(A)) \cap \hat{K}).$$

Hence, it suffices to show the existence of $C > 0$ with

$$\lambda^d(P^{-1}(B) \cap \hat{K}) \le C\lambda_P^k(B) \quad \text{for all } P \in \mathcal{P} \text{ and measurable } B \subset \operatorname{im}(P).$$

To this end, consider for every $P \in \mathcal{P}$ the linear isomorphism

$$\Phi_P : \mathbb{R}^d \to \operatorname{im}(P) \oplus_o \ker(P), \quad x \mapsto (Px, (I-P)x),$$

with inverse $\Phi_P^{-1}(u,v) = u + v$. We identify $\operatorname{im}(P) \oplus_o \ker(P)$ (up to isometry) with the Euclidean space $\mathbb{R}^d$ by defining the inner product

$$\langle (u_1, v_1), (u_2, v_2) \rangle_e := \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^d$. This gives us a norm $\|\cdot\|_e$ and a d-dimensional Lebesgue measure λ_e^d on $\operatorname{im}(P) \oplus_o \ker(P)$. For every $(u,v) \in \operatorname{im}(P) \oplus_o \ker(P) \backslash \{(0,0)\}$ we obtain

$$\begin{aligned} \left\|\Phi_P^{-1}(u,v)\right\|^2 &= \|u+v\|^2 = \|u\|^2 + 2\langle u, v\rangle + \|v\|^2 \\ &= \left(1 + 2\frac{\langle u, v\rangle}{\|u\|^2 + \|v\|^2}\right)\left(\|u\|^2 + \|v\|^2\right) \\ &= \left(1 + 2\frac{\langle u, v\rangle}{\|u\|^2 + \|v\|^2}\right)\|(u,v)\|_e^2 . \end{aligned}$$

Since $(u,v) = (Px, (I-P)x)$ with $x = u + v$, we obtain

$$\left\|\Phi_P^{-1}(u,v)\right\|^2 = \left(1 + 2\frac{\langle Px, (I-P)x\rangle}{\|Px\|^2 + \|(I-P)x\|^2}\right)\|(u,v)\|_e^2 .$$

The function $f(P,x) := \frac{\langle Px,(I-P)x\rangle}{\|Px\|^2+\|(I-P)x\|^2}$, $f : \mathcal{P} \times (\mathbb{R}^d \backslash \{0\}) \to \mathbb{R}$, does not depend on the norm of the vector x, since for every $\lambda \in \mathbb{R}\backslash\{0\}$ we have

$$\frac{\langle P(\lambda x), (I-P)(\lambda x)\rangle}{\|P(\lambda x)\|^2 + \|(I-P)(\lambda x)\|^2} = \frac{\langle Px, (I-P)x\rangle}{\|Px\|^2 + \|(I-P)x\|^2}.$$

Consequently, continuity of f and compactness of $\mathcal{P} \times S_1(0)$ implies the existence of $M \ge 0$ with $f(P,x) \le M$ for all $(P,x) \in \mathcal{P} \times (\mathbb{R}^d \backslash \{0\})$. This yields

$$\left\|\Phi_P^{-1}(u,v)\right\| \le \sqrt{1+2M}\,\|(u,v)\|_e \quad \text{for all } P \in \mathcal{P} \text{ and } (u,v) \in \operatorname{im}(P) \oplus_o \ker(P).$$

For the volume distortion $|\det \Phi_P^{-1}|$ of Φ_P^{-1} this gives us

$$\left|\det \Phi_P^{-1}\right| \le (1+2M)^{\frac{d}{2}} \quad \text{for all } P \in \mathcal{P}.$$

Using this estimate, we obtain for every measurable set $A \subset \hat{K}$

$$\begin{aligned} \lambda^d(A) &= \int_{\mathbb{R}^d} \mathbb{1}_A d\lambda^d = \int_{\operatorname{im}(P)\oplus_o \ker(P)} \mathbb{1}_A(\Phi_P^{-1}(x))|\det \Phi_P^{-1}| d\lambda_e^d(x) \\ &\le (1+2M)^{\frac{d}{2}} \int_{\operatorname{im}(P)\oplus_o \ker(P)} \mathbb{1}_{\Phi_P(A)}(x) d\lambda_e^d(x) = (1+2M)^{\frac{d}{2}} \lambda_e^d(\Phi_P(A)). \end{aligned}$$

In order to finish the proof, it suffices to show the existence of $\tilde{C} > 0$ with

$$\lambda_e^d\left(\Phi_P(P^{-1}(B) \cap \hat{K})\right) \leq \tilde{C}\lambda_P^k(B) \text{ for all } P \in \mathcal{P} \text{ and measurable } B \subset \operatorname{im}(P).$$

By the theorem of Fubini we have

$$\lambda_e^d\left(\Phi_P(P^{-1}(B) \cap \hat{K})\right) = \int_{\ker(P)} \int_{\operatorname{im}(P)} \mathbb{1}_{\Phi_P(P^{-1}(B)\cap\hat{K})}(u, v)\, du\, dv.$$

Together with

$$\begin{aligned} \Phi_P\left(P^{-1}(B) \cap \hat{K}\right) &= \left\{(Px, (I-P)x) \;:\; x \in P^{-1}(B) \cap \hat{K}\right\} \\ &\subset B \times \left(\ker(P) \cap (I-P)\hat{K}\right) \end{aligned}$$

this leads to

$$\begin{aligned} \lambda_e^d\left(\Phi_P(P^{-1}(B) \cap \hat{K})\right) &\leq \int_{\ker(P)} \int_{\operatorname{im}(P)} \mathbb{1}_{B\times(\ker(P)\cap(I-P)\hat{K})}(u, v)\, du\, dv \\ &= \int_{\ker(P)} \int_{\operatorname{im}(P)} \mathbb{1}_B(u) \cdot \mathbb{1}_{\ker(P)\cap(I-P)\hat{K}}(v)\, du\, dv \\ &= \int_{\ker(P)} \mathbb{1}_{\ker(P)\cap(I-P)\hat{K}}(v) \left(\int_{\operatorname{im}(P)} \mathbb{1}_B(u)\, du\right) dv \\ &= \lambda_P^k(B) \cdot \int_{\ker(P)} \mathbb{1}_{(I-P)\hat{K}}(v)\, dv \\ &= \lambda_P^k(B) \cdot \lambda_{I-P}^{d-k}((I-P)\hat{K}). \end{aligned}$$

Hence, it suffices to show that $\lambda_{I-P}^{d-k}((I-P)\hat{K})$ is bounded on $\mathcal{P}$. But this follows from Lemma 4.1.7. □

In the following, we use the notation $\varphi^h(t, x, u) = \Lambda_u(t, 0)x$ and $\varphi^s(t, u, v) = \int_0^t \Lambda_u(t, s)Bv(s)ds$. Then, according to (4.3), we have

$$\varphi(t, x, (u, v)) \equiv \varphi^h(t, x, u) + \varphi^s(t, u, v).$$

We assume that for the bilinear system

$$\dot{x}(t) = A(u(t))x(t), \quad u \in \mathcal{U}, \tag{4.9}$$

we have a continuous decomposition

$$\mathcal{U} \times \mathbb{R}^d = \mathcal{W}_1 \oplus \mathcal{W}_2$$

with invariant subbundles $\mathcal{W}_1$ and $\mathcal{W}_2$ of dimension d_1 and d_2, respectively ($d = d_1 + d_2$). That is,

$$\mathbb{R}^d = W_1(u) \oplus_i W_2(u) \text{ for all } u \in \mathcal{U}$$

with linear subspaces $W_1(u)$ and $W_2(u)$ ($\dim W_j(u) = d_j$, $j = 1, 2$) and

$$\mathcal{W}_j = \bigcup_{u\in\mathcal{U}} \{u\} \times W_j(u), \quad j = 1, 2.$$

Invariance of the subbundles means that $\Phi_t(\mathcal{W}_j) = \mathcal{W}_j$ $(j = 1, 2)$ for all $t \in \mathbb{R}$, where Φ denotes the control flow of system (4.9). Continuity of the decomposition means that the map $u \mapsto P(u)$, $\mathcal{U} \to \mathcal{P}(d_1, d, \mathbb{R})$, is continuous, where $\mathcal{U}$ is equipped with the weak*-topology, and $P(u)$ is the projection onto $W_1(u)$ with respect to the decomposition $\mathbb{R}^d = W_1(u) \oplus_i W_2(u)$. By Lemma A.3.9 this is no additional assumption we have to impose on the decomposition, since it is always satisfied. From the invariance of the subbundles $\mathcal{W}_j$ it follows that

$$P(\Theta_t u)\varphi^h(t, x, u) = \varphi^h(t, P(u)x, u) \text{ for all } (t, x, u) \in \mathbb{R} \times \mathbb{R}^d \times \mathcal{U}. \qquad (4.10)$$

By $\lambda_u^{d_1}$ we denote the d_1-dimensional Lebesgue measure on $W_1(u)$ for $u \in \mathcal{U}$.

The following proposition provides a lower bound for the invariance entropy $h_{\text{inv}}(K, Q)$ of the inhomogeneous bilinear system (4.1) in terms of the asymptotic behavior of a function, which describes a lower bound of the volume growth on one of the subbundles $\mathcal{W}_1$ and $\mathcal{W}_2$. The idea of the proof is essentially the same as in Theorem 3.2.1. But it is technically more complicated, since we have to consider the projection of the control system to a subbundle of $\mathcal{U} \times \mathbb{R}^d$, which itself is not a control system in the sense of our definition. Another difference is that we cannot use the Liouville Formula in order to describe the volume growth. In the subsequent theorem we will see that by choosing an appropriate volume growth function, we obtain as a lower bound of the invariance entropy the sum of the positive infimal Lyapunov exponents on the subbundles of an invariant decomposition of $\mathcal{U} \times \mathbb{R}^d$.

4.1.9 Proposition:
Consider the inhomogeneous bilinear system (4.1) *with U and V convex. Let $K, Q \subset \mathbb{R}^d$ be compact with $K \subset Q$, Q being controlled invariant, and $\lambda^d(K) > 0$. Assume that for system* (4.9) *there exists a continuous decomposition*

$$\mathcal{U} \times \mathbb{R}^d = \mathcal{W}_1 \oplus \mathcal{W}_2$$

into invariant subbundles. Let $k := \dim \mathcal{W}_1$. Further assume that there exists a function $f : \mathbb{R}_0^+ \to (0, \infty)$ with

$$\lambda^k_{\Theta_t u}(\varphi^h(t, A, u)) \geq f(t)\lambda^k_u(A) \qquad (4.11)$$

for all $t \in \mathbb{R}_0^+$, $u \in \mathcal{U}$ and for all Lebesgue measurable sets $A \subset W_1(u)$. Then the following estimate holds:

$$\boxed{h_{\text{inv}}(K, Q) \geq \limsup_{t \to \infty} \frac{1}{t} \ln f(t).} \qquad (4.12)$$

Proof:
For given $T, \varepsilon > 0$ let $\mathcal{S} = \{(u_1, v_1), \dots, (u_n, v_n)\}$ be a minimal (T, ε)-spanning set for (K, Q). Define

$$K_j := \{x \in K \mid \varphi([0, T], x, (u_j, v_j)) \subset N_\varepsilon(Q)\}, \quad j = 1, \dots, n.$$

Since $\mathcal{S}$ is (T, ε)-spanning, we have $\bigcup_{j=1}^n K_j = K$. By continuous dependence on initial conditions, K_j is open relative to K, which by Lemma 4.1.6 guarantees that the k-dimensional Lebesgue measure $\lambda^k_{u_j}(P(u_j)K_j)$ is well-defined. From the definition of K_j it immediately follows that

$$\begin{aligned} P(\Theta_T u_j)\varphi(T, K_j, (u_j, v_j)) &= P(\Theta_T u_j)\varphi^h(T, K_j, u_j) + P(\Theta_T u_j)\varphi^s(T, u_j, v_j) \\ &\subset P(\Theta_T u_j)N_\varepsilon(Q) \end{aligned}$$

or equivalently

$$P(\Theta_t u_j)\varphi^h(T, K_j, u_j) \subset P(\Theta_T u_j)N_\varepsilon(Q) - P(\Theta_T u_j)\varphi^s(T, u_j, v_j).$$

Consequently,

$$\begin{aligned} &\lambda^k_{\Theta_T u_j}\left(P(\Theta_T u_j)\varphi^h(T, K_j, u_j)\right) \\ &\qquad \le \lambda^k_{\Theta_T u_j}\left(P(\Theta_T u_j)N_\varepsilon(Q) - P(\Theta_T u_j)\varphi^s(T, u_j, v_j)\right). \end{aligned}$$

Since the Lebesgue measure is invariant under translations, it follows that

$$\lambda^k_{\Theta_T u_j}\left(P(\Theta_T u_j)\varphi^h(T, K_j, u_j)\right) \le \lambda^k_{\Theta_T u_j}\left(P(\Theta_T u_j)N_\varepsilon(Q)\right). \tag{4.13}$$

By (4.10) we obtain

$$P(\Theta_T u_j)\varphi^h(T, K_j, u_j) = \varphi^h(T, P(u_j)K_j, u_j).$$

Together with assumption (4.11) this implies

$$\lambda^k_{\Theta_T u_j}\left(P(\Theta_T u_j)\varphi^h(T, K_j, u_j)\right) \ge f(T)\lambda^k_{u_j}\left(P(u_j)K_j\right). \tag{4.14}$$

Hence, we have

$$\begin{aligned} \sum_{j=1}^n \lambda^k_{u_j}\left(P(u_j)K_j\right) &\le n \max_{j=1,\dots,n} \lambda^k_{u_j}\left(P(u_j)K_j\right) \\ &\overset{(4.14)}{\le} nf(T)^{-1} \max_{j=1,\dots,n} \lambda^k_{\Theta_T u_j}\left(P(\Theta_T u_j)\varphi^h(T, K_j, u_j)\right) \\ &\overset{(4.13)}{\le} nf(T)^{-1} \max_{j=1,\dots,n} \lambda^k_{\Theta_T u_j}\left(P(\Theta_T u_j)N_\varepsilon(Q)\right) \\ &\le nf(T)^{-1} \sup_{u\in\mathcal{U}} \lambda^k_{\Theta_T u}\left(P(\Theta_T u)N_\varepsilon(Q)\right). \end{aligned}$$

Since $N_\varepsilon(Q)$ is bounded, we can find a ball $B \subset \mathbb{R}^d$ with $N_\varepsilon(Q) \subset B$. Then

$$\begin{aligned} \sum_{j=1}^n \lambda^k_{u_j}\left(P(u_j)K_j\right) &\le nf(T)^{-1} \sup_{u\in\mathcal{U}} \lambda^k_{\Theta_T u}\left(P(\Theta_T u)B\right) \\ &= nf(T)^{-1} \sup_{u\in\mathcal{U}} \lambda^k_u\left(P(u)B\right). \end{aligned}$$

By compactness of $\mathcal{U}$ in the weak*-topology, continuity of the map $u \mapsto P(u)$ and Lemma 4.1.7 we obtain an upper bound $\alpha > 0$ for the supremum above. By Lemma 4.1.8 there exists a lower bound $\beta > 0$ for the left-hand side. Hence,

$$\beta \le nf(T)^{-1}\alpha \quad \Rightarrow \quad n \ge \frac{\beta}{\alpha} f(T). \tag{4.15}$$

Since $n = r_{\text{inv}}(T, \varepsilon, K, Q)$, for $T \to \infty$ it follows that

$$\begin{aligned} h_{\text{inv}}(K,Q) \;\geq\; h_{\text{inv}}(\varepsilon, K, Q) &\geq \limsup_{T\to\infty} \frac{1}{T} \ln\left(\frac{\beta}{\alpha} f(T)\right) \\ &= \limsup_{T\to\infty} \left(\frac{1}{T}\ln\frac{\beta}{\alpha} + \frac{1}{T}\ln f(T)\right) = \limsup_{T\to\infty} \frac{1}{T}\ln f(T), \end{aligned}$$

which is the desired inequality. □

By Colonius & Kliemann [16, Theorem 5.1.4, p. 144] there exists a unique decomposition of $\mathcal{U} \times \mathbb{R}^d$ into invariant subbundles

$$\mathcal{U} \times \mathbb{R}^d = \mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_r, \tag{4.16}$$

where $\mathcal{W}_i$ corresponds to a connected component of the chain recurrent set of the projective flow $\mathbb{P}\Phi$, where Φ is the control flow of the bilinear system (4.9).[2] We will use this decomposition in the following theorem.

4.1.10 Theorem (Lower Bound for Inhomog. Bilinear Systems): *Consider the inhomogeneous bilinear system* (4.1) *with U and V convex. Let $K, Q \subset \mathbb{R}^d$ be compact sets such that $K \subset Q$, Q is controlled invariant, and $\lambda^d(K) > 0$. Consider the decomposition* (4.16) *of $\mathcal{U} \times \mathbb{R}^d$ into the invariant subbundles $\mathcal{W}_i$. For each subbundle define the infimal Lyapunov exponent*

$$\kappa_i^* := \inf_{\substack{(u,x)\in\mathcal{W}_i \\ x\neq 0}} \limsup_{t\to\infty} \frac{1}{t} \ln \left\| \varphi^h(t,x,u) \right\|.$$

Then the following estimate holds:

$$\boxed{h_{\text{inv}}(K,Q) \geq \sum_{i:\ \kappa_i^*>0} \kappa_i^* \dim \mathcal{W}_i.} \tag{4.17}$$

Proof:
We may assume that $\kappa_1^*, \ldots, \kappa_l^* > 0$ and $\kappa_{l+1}^*, \ldots, \kappa_r^* \leq 0$ for some $0 \leq l \leq r$. Let $k_i := \dim \mathcal{W}_i$ for $i = 1, \ldots, r$. The proof now proceeds in three steps.

<u>Step 1:</u> Let $f_1, \ldots, f_l : \mathbb{R}_0^+ \to (0, \infty)$ be functions satisfying

$$\lambda_{\Theta_t u}^{k_i}(\varphi^h(t,A,u)) \geq f_i(t) \lambda_u^{k_i}(A)$$

for all $t \in \mathbb{R}_0^+$, $u \in \mathcal{U}$ and for all Lebesgue measurable sets $A \subset W_i(u)$. We want to show that this implies

$$h_{\text{inv}}(K,Q) \geq \limsup_{t\to\infty} \frac{1}{t} \ln \prod_{i=1}^{l} f_i(t). \tag{4.18}$$

[2] The existence of the decomposition actually follows from *Selgrade's Theorem* on the existence of finest Morse decompositions for linear flows on vector bundles with chain transitive base flow. See also Colonius & Kliemann [16, Theorem 5.2.5, p. 152].

Without loss of generality we may assume that $l = 2$: Let $\mathcal{W} := \mathcal{W}_1 \oplus \mathcal{W}_2$. Fix a control function $u \in \mathcal{U}$. Let $A \subset W(u)$ denote a Lebesgue measurable set which is the product of two Lebesgue measurable sets $A_1 \subset W_1(u)$ and $A_2 \subset W_2(u)$,

$$A = \{a_1 + a_2 \mid (a_1, a_2) \in A_1 \times A_2\}.$$

For fixed $t \geq 0$ we consider the linear transformation

$$g_{\Theta_t u} : W(\Theta_t u) \to W_1(\Theta_t u) \oplus_o W_2(\Theta_t u), \quad g_{\Theta_t u}(x) = (x_1, x_2),$$

which splits a point $x \in W(\Theta_t u)$ into its components $x_1 \in W_1(\Theta_t u)$ and $x_2 \in W_2(\Theta_t u)$. On the external sum $W_1(\Theta_t u) \oplus_o W_2(\Theta_t u)$ we consider an inner product $\langle \cdot, \cdot \rangle_{\Theta_t u}$ with the property that $W_1(\Theta_t u) = W_2(\Theta_t u)^\perp$. The k-dimensional Lebesgue measure induced by $\langle \cdot, \cdot \rangle_{\Theta_t u}$ is denoted by $\lambda^k_{\Theta_t u, o}$. Using the transformation theorem and the theorem of Fubini we obtain

$$\begin{aligned}
&\lambda^k_{\Theta_t u}\left(\varphi^h(t, A, u)\right) = \int_{\varphi^h(t,A,u)} d\lambda^k_{\Theta_t u} = \int_{g_{\Theta_t u}(\varphi^h(t,A,u))} \left|\det g^{-1}_{\Theta_t u}\right| d\lambda^k_{\Theta_t u, o} \\
&= \left|\det g^{-1}_{\Theta_t u}\right| \int_{W_1(\Theta_t u) \oplus_o W_2(\Theta_t u)} \mathbb{1}_{g_{\Theta_t u}(\varphi^h(t,A,u))}(x_1, x_2) d\lambda^k_{\Theta_t u, o}(x_1, x_2) \\
&= \left|\det g^{-1}_{\Theta_t u}\right| \int_{W_1(\Theta_t u)} \int_{W_2(\Theta_t u)} \mathbb{1}_{g_{\Theta_t u}(\varphi^h(t,A,u))}(x_1, x_2) d\lambda^{k_2}_{\Theta_t u}(x_2) d\lambda^{k_1}_{\Theta_t u}(x_1).
\end{aligned}$$

Since $\varphi^h(t, \cdot, u)$ is linear, we have

$$\begin{aligned}
\varphi^h(t, A, u) &= \left\{\varphi^h(t, a_1, u) + \varphi^h(t, a_2, u) \;:\; (a_1, a_2) \in A_1 \times A_2\right\} \\
&= \left\{x_1 + x_2 \;:\; (x_1, x_2) \in \varphi^h(t, A_1, u) \times \varphi^h(t, A_2, u)\right\}
\end{aligned}$$

and thus

$$g_{\Theta_t u}\left(\varphi^h(t, A, u)\right) = \varphi^h(t, A_1, u) \times \varphi^h(t, A_2, u).$$

This implies

$$\begin{aligned}
&\lambda^k_{\Theta_t u}\left(\varphi^h(t, A, u)\right) \\
&= \left|\det g^{-1}_{\Theta_t u}\right| \int_{W_1(\Theta_t u)} \int_{W_2(\Theta_t u)} \mathbb{1}_{\varphi^h(t,A_1,u)}(x_1) \mathbb{1}_{\varphi^h(t,A_2,u)}(x_2) d\lambda^{k_2}_{\Theta_t u}(x_2) d\lambda^{k_1}_{\Theta_t u}(x_1) \\
&= \left|\det g^{-1}_{\Theta_t u}\right| \lambda^{k_1}_{\Theta_t u}\left(\varphi^h(t, A_1, u)\right) \lambda^{k_2}_{\Theta_t u}\left(\varphi^h(t, A_2, u)\right) \\
&\geq \left|\det g^{-1}_{\Theta_t u}\right| \left(f_1(t) \lambda^{k_1}_u(A_1)\right) \left(f_2(t) \lambda^{k_2}_u(A_2)\right).
\end{aligned}$$

Further we have

$$\begin{aligned}
\lambda^{k_1}_u(A_1) \lambda^{k_2}_u(A_2) &= \int_{W_1(u)} \int_{W_2(u)} \mathbb{1}_{A_1}(x_1) \mathbb{1}_{A_2}(x_2) d\lambda^{k_2}_u(x_2) d\lambda^{k_1}_u(x_1) \\
&= \int_{W_1(u) \oplus_o W_2(u)} \mathbb{1}_{A_1 \times A_2}(x) d\lambda^k_{u,o}(x) \\
&= \int_{g_u^{-1}(W_1(u) \oplus_o W_2(u))} \mathbb{1}_{A_1 \times A_2}(g_u(x)) |\det g_u| d\lambda^k_u(x)
\end{aligned}$$

$$= |\det g_u| \int_{W(u)} \mathbb{1}_{g_u^{-1}(A_1 \times A_2)}(x) d\lambda_u^k(x)$$
$$= |\det g_u| \int_{W(u)} \mathbb{1}_A(x) d\lambda_u^k(x) = |\det g_u| \lambda_u^k(A).$$

Consequently, we obtain

$$\begin{aligned} \lambda_{\Theta_t u}^k \left(\varphi^h(t, A, u)\right) &\geq |\det g_u| \left|\det g_{\Theta_t u}^{-1}\right| f_1(t) f_2(t) \lambda_u^k(A) \\ &\geq \left(\inf_{u_1 \in \mathcal{U}} |\det g_{u_1}|\right) \left(\inf_{u_2 \in \mathcal{U}} \left|\det g_{u_2}^{-1}\right|\right) f_1(t) f_2(t) \lambda_u^k(A). \end{aligned}$$

With the same arguments used in the proof of Lemma 4.1.8 one can show that $\inf_{u_1} |\det g_{u_1}|$ and $\inf_{u_2} |\det g_{u_2}^{-1}|$ are positive. Thus, there exists $C > 0$ with

$$\lambda_{\Theta_t u}^k \left(\varphi^h(t, A, u)\right) \geq C f_1(t) f_2(t) \lambda_u^k(A) \quad \text{for all } u \in \mathcal{U},\ t \geq 0,$$

and all measurable sets $A \subset W(u)$, which can be written as products. From the linearity of the maps $\varphi^h(t, \cdot, u)$ it follows that the same inequality holds for all measurable sets $A \subset W(u)$ and hence (4.11) is satisfied with $f(t) := C f_1(t) f_2(t)$. This implies

$$h_{\text{inv}}(K, Q) \geq \limsup_{t \to \infty} \frac{1}{t} \ln \left(C f_1(t) f_2(t)\right) = \limsup_{t \to \infty} \frac{1}{t} \ln \left(f_1(t) f_2(t)\right),$$

which proves (4.18).

<u>Step 2:</u> For every $i \in \{1, \ldots, r\}$ we define

$$f_i(t) := \left[\inf_{\substack{(u,x) \in \mathcal{W}_i \\ \|x\|=1}} \left\|\varphi^h(t, x, u)\right\|\right]^{k_i}, \quad f_i : \mathbb{R}_0^+ \to (0, \infty),$$

and

$$\xi_{\min}(\mathcal{W}_i) := \inf_{\substack{(u,x) \in \mathcal{W}_i \\ x \neq 0}} \liminf_{t \to -\infty} \frac{1}{t} \ln \left\|\varphi^h(t, y, u)\right\|.$$

We will first show that $h_{\text{inv}}(K, Q)$ is bounded from below by $\sum_{i=1}^l k_i \xi_{\min}(\mathcal{W}_i)$ and then, that $\xi_{\min}(\mathcal{W}_i) = \kappa_i^*$, which completes the proof. Note that f_i satisfies (4.11) on the subbundle $\mathcal{W}_i$, since for all $(u, x) \in \mathcal{W}_i$ and $t \geq 0$ we have[3]

$$\left\|\varphi^h(t, x, u)\right\| = \left\|\varphi^h\left(t, \tfrac{x}{\|x\|}, u\right)\right\| \cdot \|x\| \geq f_i(t)^{1/k_i} \|x\|.$$

Now let $g_i(t) := \inf_{\substack{(u,x) \in \mathcal{W}_i \\ \|x\|=1}} \frac{1}{t} \ln \|\varphi^h(t, x, u)\|$ for all $t > 0$ and $i = 1, \ldots, l$. Then

$$\exp\left(t k_i g_i(t)\right) \equiv f_i(t), \quad i = 1, \ldots, l, \tag{4.19}$$

[3] In general, for a linear map $L : V \to W$ between n-dimensional Euclidean vector spaces, the inequality $\|Lv\|_W \geq c\|v\|_V$ for all $v \in V$ with a constant $c > 0$ implies that $\lambda_W^n(L(A)) \geq c^n \lambda_V^n(A)$ for all measurable sets $A \subset V$. This follows, e.g., from Boichenko & Leonov & Reitmann [8, Proposition 7.2.1, p. 73].

and

$$\begin{aligned}
g_i(t) &= \frac{1}{t}\ln\left(\inf_{\substack{(u,x)\in\mathcal{W}_i\\ \|x\|=1}}\left\|\varphi^h(t,x,u)\right\|\right) = \frac{1}{t}\ln\left(\inf_{\substack{(u,x)\in\mathcal{W}_i\\ x\neq 0}}\frac{\|\varphi^h_{t,u}(x)\|}{\|x\|}\right)\\
&= \frac{1}{t}\ln\left(\inf_{\substack{(\Theta_t u,y)\in\mathcal{W}_i\\ y\neq 0}}\frac{\|y\|}{\|(\varphi^h_{t,u})^{-1}(y)\|}\right) = \frac{1}{t}\ln\left(\inf_{\substack{(v,y)\in\mathcal{W}_i\\ y\neq 0}}\frac{\|y\|}{\|(\varphi^h_{t,\Theta_{-t}v})^{-1}(y)\|}\right)\\
&= \frac{1}{t}\ln\left(\sup_{\substack{(v,y)\in\mathcal{W}_i\\ y\neq 0}}\frac{\|(\varphi^h_{t,\Theta_{-t}v})^{-1}(y)\|}{\|y\|}\right)^{-1}\\
&= -\frac{1}{t}\ln\left(\sup_{\substack{(v,y)\in\mathcal{W}_i\\ \|y\|=1}}\left\|(\varphi^h_{t,\Theta_{-t}v})^{-1}(y)\right\|\right) = -\frac{1}{t}\sup_{\substack{(v,y)\in\mathcal{W}_i\\ \|y\|=1}}\ln\left\|(\varphi^h_{t,\Theta_{-t}v})^{-1}(y)\right\|.
\end{aligned}$$

Note that $(\varphi^h_{t,\Theta_{-t}v})^{-1}\equiv\varphi^h_{-t,v}$ (see Corollary 1.2.12). Hence, we obtain

$$\begin{aligned}
\liminf_{t\to\infty} g_i(t) &= \liminf_{t\to\infty}\left(-\frac{1}{t}\sup_{\substack{(v,y)\in\mathcal{W}_i\\ \|y\|=1}}\ln\left\|\varphi^h(-t,y,v)\right\|\right)\\
&= -\limsup_{t\to\infty}\frac{1}{t}\sup_{\substack{(v,y)\in\mathcal{W}_i\\ \|y\|=1}}\ln\left\|\varphi^h(-t,y,v)\right\|.
\end{aligned}$$

By Colonius & Kliemann [16, Proposition 5.4.15, p. 178][4], applied to the flow $(t,(y,v))\mapsto(\Theta_{-t}v,\varphi^h(-t,y,v))$, $\mathbb{R}\times\mathcal{W}_i\to\mathcal{W}_i$, we get

$$\begin{aligned}
\liminf_{t\to\infty} g_i(t) &= -\sup_{\substack{(v,y)\in\mathcal{W}_i\\ y\neq 0}}\limsup_{t\to\infty}\frac{1}{t}\ln\left\|\varphi^h(-t,y,v)\right\|\\
&= \inf_{\substack{(v,y)\in\mathcal{W}_i\\ y\neq 0}}\liminf_{t\to\infty}\frac{1}{(-t)}\ln\left\|\varphi^h(-t,y,v)\right\|\\
&= \inf_{\substack{(v,y)\in\mathcal{W}_i\\ \|y\|=1}}\liminf_{t\to-\infty}\frac{1}{t}\ln\left\|\varphi^h(t,y,v)\right\| = \xi_{\min}(\mathcal{W}_i).
\end{aligned}$$

Together with (4.18) we obtain

$$\begin{aligned}
h_{\mathrm{inv}}(K,Q) &\geq \limsup_{t\to\infty}\frac{1}{t}\ln\prod_{i=1}^{l}f_i(t) = \limsup_{t\to\infty}\frac{1}{t}\sum_{i=1}^{l}\ln f_i(t)\\
&\geq \liminf_{t\to\infty}\frac{1}{t}\sum_{i=1}^{l}\ln f_i(t) \geq \sum_{i=1}^{l}\liminf_{t\to\infty}\frac{1}{t}\ln f_i(t)\\
&\overset{(4.19)}{=} \sum_{i=1}^{l}\liminf_{t\to\infty}\frac{1}{t}\ln\exp(tk_i g_i(t)) = \sum_{i=1}^{l}k_i\liminf_{t\to\infty}g_i(t)\\
&= \sum_{i=1}^{l}k_i\xi_{\min}(\mathcal{W}_i).
\end{aligned}$$

[4]The proposition says that for a linear flow Φ on a vector bundle $\pi:\mathcal{V}\to B$ the numbers $\sup_{v\in\mathcal{V},\|v\|\neq 0}\limsup_{t\to\infty}\frac{1}{t}\ln\|\Phi_t v\|$ and $\limsup_{t\to\infty}\frac{1}{t}\sup_{b\in B}\ln\|\Phi(t,b)\|$ coincide.

Step 3: We prove that $\xi_{\min}(\mathcal{W}_i) = \kappa_i^*$, which yields the assertion: By Colonius & Kliemann [16, Theorem 5.1.6, p. 146] the infimal Lyapunov exponent κ_i^* equals the infimum of the Morse spectrum of the control flow Φ on $\mathcal{W}_i$:[5]

$$\kappa_i^* = \inf \Sigma_{\mathrm{Mo}}(\Phi|_{\mathcal{W}_i}).$$

By Colonius & Kliemann [16, Proposition 5.3.4, p. 161] we have

$$\inf \Sigma_{\mathrm{Mo}}(\Phi|_{\mathcal{W}_i}) = -\sup \Sigma_{\mathrm{Mo}}(\Phi^*|_{\mathcal{W}_i}),$$

where Φ^* is the time-reversed control flow $(t,(u,x)) \mapsto (\theta_{-t}u, \varphi^h(-t,x,u))$. Again, by Colonius & Kliemann [16, Theorem 5.1.6, p. 146] the supremum of the Morse spectrum of $\Phi^*|_{\mathcal{W}_i}$ equals the maximal Lyapunov exponent. Hence,

$$\begin{aligned}
\kappa_i^* &= -\sup \Sigma_{\mathrm{Mo}}(\Phi^*|_{\mathcal{W}_i}) = -\sup_{\substack{(u,x)\in\mathcal{W}_i \\ x\neq 0}} \limsup_{t\to\infty} \frac{1}{t} \ln \left\|\varphi^h(-t,x,u)\right\| \\
&= \inf_{\substack{(u,x)\in\mathcal{W}_i \\ x\neq 0}} \liminf_{t\to\infty} \frac{1}{(-t)} \ln \left\|\varphi^h(-t,x,u)\right\| \\
&= \inf_{\substack{(u,x)\in\mathcal{W}_i \\ x\neq 0}} \liminf_{t\to-\infty} \frac{1}{t} \ln \left\|\varphi^h(t,x,u)\right\| = \xi_{\min}(\mathcal{W}_i).
\end{aligned}$$

This finishes the proof. □

4.1.11 Remark:
Note that for a linear control system the right-hand side in Formula (4.17) reduces to the sum of the eigenvalues of A with positive real parts, counted with multiplicities, since the projective linear flow $(t, \mathbb{P}x) \mapsto \mathbb{P}e^{At}x$ has exactly r chain recurrent components $\mathcal{M}_1, \ldots, \mathcal{M}_r$, where r is the number of different Lyapunov exponents of the linear flow $e^{At}x$, i.e., the number of different real parts. Moreover, $\mathbb{P}^{-1}\mathcal{M}_i$ is the i^{th} Lyapunov space $L_i(A)$ of A (see Colonius [14, Theorem 5.2]). In this case, the decomposition (4.16) is given by

$$\mathcal{U} \times \mathbb{R}^d = (\mathcal{U} \times L_1(A)) \oplus \cdots \oplus (\mathcal{U} \times L_r(A)).$$

The next proposition yields an upper bound for the invariance entropy of an inhomogeneous bilinear system. It is an easy consequence of Theorem 4.1.2 and hence we omit its proof.

4.1.12 Proposition:
Consider the inhomogeneous bilinear system (4.1)*. Let $K, Q \subset \mathbb{R}^d$ be compact with $K \subset Q$ and Q being controlled invariant. Assume that $u^0 \in \mathcal{U}$ is a constant control function such that Q is controlled invariant with respect to the linear system*

$$\dot{x}(t) = \left(A_0 + \sum_{i=1}^{m_1} u_i^0 A_i\right) x(t) + Bv(t), \quad v \in \mathcal{V}.$$

[5] Note that by Proposition 1.3.15 the base flow $\Theta : \mathbb{R} \times \mathcal{U} \to \mathcal{U}$ is chain transitive and hence chain recurrent.

Then Q is also controlled invariant with respect to system (4.1) *with*

$$\boxed{h^{+}_{\mathrm{inv}}(K,Q) \leq \sum_{i:\ \mathrm{Re}(\lambda_i(u^0))>0} \mathrm{Re}(\lambda_i(u^0)),}$$

where $\lambda_1(u^0),\dots,\lambda_d(u^0)$ denote the eigenvalues of $A_0 + \sum_{i=1}^{m_1} u_i^0 A_i$.

4.1.13 Example:
Consider the one-dimensional system

$$\dot{x}(t) = (a+u(t))x(t) + v(t), \quad u(t) \in [u_{\min}, u_{\max}], \quad v(t) \in [v_{\min}, v_{\max}].$$

Assume that $v_{\min} < v_{\max}$ and $a + u_{\min} > 0$. Let $Q := \frac{1}{a+u_{\min}}[-v_{\max}, -v_{\min}]$. Then Q is controlled invariant, since for every $x \in Q$ we have

$$v_x := -(a+u_{\min})x \in [v_{\min}, v_{\max}] \ \text{ and } \ (a+u_{\min}) + v_x = 0.$$

From Proposition 4.1.4(ii) and Proposition 4.1.12 we obtain

$$h_{\mathrm{inv}}(Q) = a + u_{\min}.$$

This shows that the lower bound of Theorem 4.1.10 can also be tight for inhomogeneous bilinear systems which are not linear. ◊

We end this section with a geometric criterion for finiteness of $h^*_{\mathrm{inv}}(Q)$.

4.1.14 Proposition:
Let $Q \subset \mathbb{R}^d$ be compact and convex with $\mathrm{int}\, Q \neq \emptyset$, and controlled invariant for the inhomogeneous bilinear system (4.1)*. Then the following statements are equivalent:*

(i) $h^*_{\mathrm{inv}}(Q) < \infty$.

(ii) *For some $\tau_0 > 0$ there exists a finite τ_0-spanning set for* $(\mathrm{bd}\, Q, Q)$.

Proof:
The implication "(i) ⇒ (ii)" follows from Corollary 2.1.11 and Proposition 2.1.8(iii). In order to show that (ii) implies (i), let $\mathcal{S}^* = \{u_1,\dots,u_n\}$ be a finite τ_0-spanning set for $(\mathrm{bd}\, Q, Q)$ and define

$$\widehat{Q}_j := \{z \in \mathrm{bd}\, Q \mid \varphi([0,\tau_0], z, u_j) \subset Q\}, \quad j = 1,\dots,n.$$

Then $\{\widehat{Q}_j\}_{1\leq j\leq n}$ is a covering of $\mathrm{bd}\, Q$. Now fix an arbitrary $x_0 \in \mathrm{int}\, Q$ and define[6]

$$Q_j := \bigcup_{z\in\widehat{Q}_j} [x_0, z], \quad j = 1,\dots,n.$$

Since Q is convex and $\mathrm{bd}\, Q \subset Q$, we have $Q_j \subset Q$ for all $j \in \{1,\dots,n\}$. On the other hand, for every $y \in Q$ either $y = x_0$ holds or there exists a unique line

[6] By $[x,y]$ we denote the line segment $\{ty + (1-t)x \mid t \in [0,1]\}$ for $x, y \in \mathbb{R}^d$.

$g \subset \mathbb{R}^d$ such that $x_0, y \in g$. Since Q is bounded, there exists some $z \in g \cap \operatorname{bd} Q$ such that $y \in [x_0, z]$. Hence, $\{Q_j\}_{1 \leq j \leq n}$ is a covering of Q. Since $x_0 \in \operatorname{int} Q$, by continuity of $\varphi(\cdot, x, u)$ for every $j \in \{1, \ldots, n\}$ there exists a time $\tau_j > 0$ with $\varphi([0, \tau_j], x_0, u_j) \subset Q$. Let $T := \min\{\tau_0, \tau_1, \ldots, \tau_n\}$ and pick an arbitrary $x \in Q$. Then $x \in Q_j$ for some $j \in \{1, \ldots, n\}$ and hence $x \in [x_0, z]$ for some $z \in \widehat{Q}_j$. By linearity of $\varphi(t, \cdot, u)$ and convexity of Q we have

$$\varphi(t, x, u_j) \in \varphi(t, [x_0, z], u_j) = [\varphi(t, x_0, u_j), \varphi(t, z, u_j)] \subset Q \quad \text{for all } t \in [0, T].$$

Hence, $\mathcal{S}^*$ is a finite T-spanning set for Q. By Corollary 2.1.11 this implies (i). □

4.1.15 Open Question:
What is the exact value of the invariance entropy for inhomogeneous bilinear systems?

4.2 Control Sets of Control-Affine Systems

Throughout this section, we consider only control-affine systems with compact and convex control range. For this class of systems we know that the set $\mathcal{U}$ of admissible control functions becomes a compact metrizable space with the weak*-topology and the cocycle $\varphi : \mathbb{R} \times M \times \mathcal{U} \to M$ is continuous (see Proposition 1.3.14). We also know that the closure of a control set D is controlled invariant (see Lemma 2.3.1). Hence, if $\operatorname{cl} D$ is compact, we can consider the invariance entropy $h_{\text{inv}}(K, \operatorname{cl} D)$ for compact sets $K \subset D$. The following theorem shows in particular that this quantity does not depend on K, provided that K has nonvoid interior. This observation will be the key for a couple of further results relating the invariance entropy to the positive Lyapunov exponents of the given control system.

4.2.1 Theorem (Invariance Entropy of Control Sets):
Let D be a control set of the control-affine system (1.34) *with compact closure $Q := \operatorname{cl} D$ and nonvoid interior. Then the following statements hold:*

(i) *If $K_1, K_2 \subset D$ are compact sets with nonvoid interior, then*

$$\boxed{h_{\text{inv}}(K_1, Q) = h_{\text{inv}}(K_2, Q) \quad \text{and} \quad h^*_{\text{inv}}(K_1, Q) = h^*_{\text{inv}}(K_2, Q).}$$

(ii) *If local accessibility holds on Q, then $h^*_{\text{inv}}(K, Q) < \infty$ for all compact sets $K \subset D$.*

(iii) *Assume that there exists a control function $u^* \in \mathcal{U}$ and an open set $V \subset D$ such that $\pi_M \omega(x, u^*) \cap \operatorname{int} D \neq \emptyset$ for all $x \in V$. Then $h^*_{\text{inv}}(K, Q) = 0$ for all compact sets $K \subset D$.*

Proof:

(i) Consider the first equality. Obviously, it suffices to prove the inequality "$\leq$". Since $D \subset \operatorname{cl} \mathcal{O}^+(x)$ for every $x \in D$, we can assign to each $x \in K_1$ a control function $u_x \in \mathcal{U}$ and a time $t_x \geq 0$ such that $\varphi(t_x, x, u_x) \in \operatorname{int} K_2$. Trajectories cannot leave the control set and return and hence we also have $\varphi([0, t_x], x, u_x) \subset D$ (see Remark 1.3.9). By continuous dependence on initial conditions one finds a neighborhood V_x of x with $\varphi(t_x, V_x, u_x) \subset \operatorname{int} K_2$. The family $\{V_x\}_{x \in K_1}$ is an open cover of K_1. By compactness there exist $x_1, \ldots, x_n \in K_1$ with $K_1 \subset \bigcup_{i=1}^n V_{x_i}$. Now, for arbitrary $T, \varepsilon > 0$ let $\mathcal{S} = \{v_1, \ldots, v_k\}$ be a minimal (T, ε)-spanning set for (K_2, Q). For every index pair (i, j) with $1 \leq i \leq n$ and $1 \leq j \leq k$ such that there exists $x \in K_1$ with $y_x := \varphi(t_{x_i}, x, u_{x_i}) \in \operatorname{int} K_2$ and $\varphi([0, T], y_x, v_j) \subset N_\varepsilon(Q)$ we define the control function w_{ij} by

$$w_{ij}(t) := \begin{cases} u_{x_i}(t) & \text{for } t \in [0, t_{x_i}], \\ v_j(t - t_{x_i}) & \text{for } t > t_{x_i}. \end{cases}$$

The number of these control functions is bounded from above by $nk = n r_{\text{inv}}(T, \varepsilon, K_2, Q)$. Consider the set $\hat{\mathcal{S}}$ consisting of the control functions w_{ij}. Let $\hat{T} := T + \min_{i=1,\ldots,n} t_{x_i}$. Then, by construction, $\hat{\mathcal{S}}$ is a $(\hat{T}, \varepsilon)$-spanning set for (K_1, Q). Consequently,

$$r_{\text{inv}}(T, \varepsilon, K_1, Q) \leq r_{\text{inv}}(\hat{T}, \varepsilon, K_1, Q) \leq n r_{\text{inv}}(T, \varepsilon, K_2, Q).$$

Hence, we obtain

$$h_{\text{inv}}(\varepsilon, K_1, Q) \leq \limsup_{T \to \infty} \left(\frac{\ln n}{T} + \frac{\ln r_{\text{inv}}(T, \varepsilon, K_2, Q)}{T} \right) = h_{\text{inv}}(\varepsilon, K_2, Q).$$

For $\varepsilon \searrow 0$ the desired inequality $h_{\text{inv}}(K_1, Q) \leq h_{\text{inv}}(K_2, Q)$ follows. The proof for the strict invariance entropy works analogously.

(ii) Any compact subset of D is contained in a compact subset with nonvoid interior. Hence, by Proposition 2.1.8(iii), we may assume that K has nonvoid interior. By Colonius & Kliemann [16, Proposition 4.3.3(i), p. 100] the periodic points of the control flow are dense in the lift $\mathcal{D}$ of D.[7] Hence, we can find a pair $(x^*, u^*) \in \operatorname{int} D \times \mathcal{U}$ and a time $T^* > 0$ with $\varphi(T^*, x^*, u^*) = x^*$ and $\Theta_{T^*} u^* = u^*$. Since by Remark 1.3.9 trajectories cannot leave a control set and return, we have $\varphi(\mathbb{R}_0^+, x^*, u^*) \subset D$. Now assume to the contrary that $\varphi(t, x^*, u^*) \in \operatorname{bd} D$ for some $t \in (0, T^*)$. Then, by Colonius & Kliemann [16, Proposition 3.2.25, p. 66], $\varphi(t, x^*, u^*) \in \Gamma^*(D)$ (the entrance boundary of D), since $D \cap \operatorname{bd} D = \Gamma^*(D)$. But otherwise, since the point $x^* \in \operatorname{int} D$ can be steered to $\varphi(t, x^*, u^*)$, we have $\varphi(t, x^*, u^*) \in \Gamma(D)$ (the exit boundary of D). Since $\Gamma^*(D) \cap \Gamma(D) = \emptyset$, this is not possible and thus $\varphi(\mathbb{R}_0^+, x^*, u^*) \subset \operatorname{int} D$. Since $\varphi(\mathbb{R}_0^+, x^*, u^*) = \varphi([0, T^*], x^*, u^*)$ is compact, we find a compact set $\tilde{K} \subset \operatorname{int} D$ with nonvoid interior and $\varphi(\mathbb{R}_0^+, x^*, u^*) \subset \operatorname{int} \tilde{K}$. By statement (i), we may assume that $K = \tilde{K}$.

For every $x \in K \subset \operatorname{int} D$ we can find a control function $u_x \in \mathcal{U}$ and a time $t_x \geq 0$ with $\varphi(t_x, x, u_x) = x^*$ by exact controllability in the interior of control sets (see Proposition 1.3.6(iii)). By Colonius & Kliemann [16,

[7] Here, the lift of D is defined by $\mathcal{D} = \operatorname{cl}\{(u, x) \in \mathcal{U} \times M \mid \varphi(\mathbb{R}, x, u) \subset \operatorname{int} D\}$.

Lemma 3.2.21, p. 65] we may assume that $t_x \leq T_0$ for all $x \in K$ for some $T_0 > 0$.[8] By switching to the control function u^* after time t_x we can assume that

$$y_x := \varphi(T_0, x, u_x) \in \operatorname{int} K \quad \text{for all } x \in K.$$

Let V_x be a neighborhood of y_x with $V_x \subset \operatorname{int} K$. By continuity there exists a neighborhood W_x of x with $\varphi(T_0, W_x, u_x) \subset V_x \subset \operatorname{int} K$. Since $\{W_x\}_{x \in K}$ covers the compact set K, we find $x_1, \dots, x_n \in K$ ($n \in \mathbb{N}$) with $K \subset \bigcup_{j=1}^n W_{x_j}$. Consequently, the set $\mathcal{S}^* := \{u_{x_1}, \dots, u_{x_n}\}$ is T_0-spanning for (K, Q) (since trajectories starting in D cannot leave D and return). Obviously, one can construct (kT_0)-spanning sets $\mathcal{S}_k^*$ for all $k \in \mathbb{N}$ from $\mathcal{S}^*$ such that $\#\mathcal{S}_k^* \leq n^k$. This proves that $h_{\text{inv}}^*(K, Q) \leq \frac{\ln n}{T_0} < \infty$.

(iii) By property (ii) of control sets, for every $x \in K$ there exist $u_x \in \mathcal{U}$ and $t_x > 0$ with $\varphi(t_x, x, u_x) \in \operatorname{int} V$. By continuous dependence on the initial value there exists a neighborhood W_x of x with $\varphi(t_x, W_x, u_x) \subset \operatorname{int} V$. Since K is compact, finitely many of these neighborhoods are sufficient to cover K, say $W_{x_1}, \dots, W_{x_n}$. We define n control functions $v_1, \dots, v_n$ by

$$v_i(t) := \begin{cases} u_{x_i}(t) & \text{for } t \in [0, t_{x_i}], \\ u^*(t - t_{x_i}) & \text{for } t > t_{x_i}. \end{cases}$$

Then, for every $x \in K$ there exists $i \in \{1, \dots, n\}$ with $\pi_M \omega(x, v_i) \cap \operatorname{int} V \neq \emptyset$. Hence, there exists a sequence $t_n \to \infty$, $t_n > 0$, such that $\varphi(t_n, x, v_i) \in \operatorname{int} D$ for all $n \in \mathbb{N}$. Since trajectories cannot leave the control set D and return, this implies $\varphi(\mathbb{R}_0^+, x, v_i) \subset D$. It follows that $r_{\text{inv}}^*(T, K, Q) \leq n$ for all $T > 0$ and thus $h_{\text{inv}}^*(K, Q) = 0$.

□

4.2.2 Remarks:

- If the set K is only contained in Q and not in D, then the proof of the preceding theorem does not work. Hence, in particular the case $K = Q$ is excluded here, except for the trivial case of an invariant control set (where the invariance entropy vanishes).
- Note that the hypothesis of Theorem 4.2.1(iii) is in particular satisfied if there exists a point in the interior of D, which is an attractive equilibrium for some constant control function.
- If D is a control set with nonvoid interior and compact closure, then by Theorem 4.2.1 it is justified to speak of the *(strict) invariance entropy of* D, meaning the value of $h_{\text{inv}}^{(*)}(K, \operatorname{cl} D)$, where $K \subset D$ is any compact set with $\operatorname{int} K \neq \emptyset$.

[8] The statement of this lemma is the following: For a control set D with nonvoid interior and two compact sets K_1 in the domain of attraction of D and K_2 in $\operatorname{int} D$, under the assumption of local accessibility on K_2, there exists a time $T = T(K_1, K_2) < \infty$ such that $\inf\{t \geq 0 \mid \exists u : \varphi(t, x, u) = y\} \leq T$ for all $x \in K_1$ and $y \in K_2$.

It is clear that for an invariant control set D one has $h^*_{\mathrm{inv}}(K, \mathrm{cl}\, D) = 0$ for all $K \subset D$. The following example shows that the converse is in general not true.

4.2.3 Example:
Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1-function with $f(x) > 0$ for all $x \in (-1, 1)$ and $f(x) = 0$ for all $x \in \mathbb{R}\backslash(-1, 1)$. Consider the control-affine system

$$\dot{x}(t) = u_1(t) f(x(t)) + u_2(t), \quad (u_1, u_2) \in \mathcal{U},$$

with control range $U = [-1, 1] \times [0, 1]$. We want to show that $D := [-1, 1)$ is a variant control set with $h^*_{\mathrm{inv}}(K, \mathrm{cl}\, D) = 0$ for all compact sets $K \subset D$:
By setting $u_1(t) := u_2(t) := 0$ for all $t \geq 0$ one achieves that $\varphi(t, x, (u_1, u_2)) = x$ for all $x \in \mathbb{R}$ and $t \geq 0$. Consequently, every subset of $\mathbb{R}$ is controlled invariant, so in particular the set D. By this observation it is also clear that $h^*_{\mathrm{inv}}(K, \mathrm{cl}\, D) = 0$, since a single control function is sufficient to keep every point in $\mathrm{cl}\, D$ for all times. By choosing $u_2(t) := 0$ for all $t \geq 0$ and giving u_1 a constant value between -1 and 1 it is possible to steer every point in $(-1, 1)$ to the left and to the right and reach every point in $(-1, 1)$ from every other point. By choosing $u_1(t) := 0$ and $u_2(t) := 1$ for all $t \geq 0$ it is further possible to reach from any point $x \in \mathbb{R}$ every other point $y \in \mathbb{R}$ with $y > x$. Consequently, we have $D \subset \mathrm{cl}\, \mathcal{O}^+(x)$ for all $x \in [-1, 1) = D$. Since one cannot steer points $x \in \mathbb{R}\backslash D$ to the left, D is maximal with the properties (i) and (ii) of control sets, and hence is a control set (which is obviously variant). $\Diamond$

Now consider again the linear control system (4.2) with compact and convex control range U such that $0 \in \mathrm{int}\, U$. Suppose that the *Kalman rank condition* holds, i.e., that the matrix

$$[B|AB|A^2B|\cdots|A^{d-1}B] \in \mathbb{R}^{d\times(md)} \tag{4.20}$$

has full rank (cf. Sontag [50, Section 3.2]). Then there exists a unique control set $D \subset \mathbb{R}^d$ with nonvoid interior, which is bounded if the matrix A is hyperbolic, i.e., if all eigenvalues of A have nonzero real parts (see Colonius & Kliemann [16, Example 3.2.16, pp. 61–63] or Colonius & Spadini [17, Theorem 4.1]). The following theorem shows that in this case $h^*_{\mathrm{inv}}(K, \mathrm{cl}\, D) = h_{\mathrm{inv}}(K, \mathrm{cl}\, D)$ for all compact sets $K \subset D$ with $\mathrm{int}\, K \neq \emptyset$.

4.2.4 Theorem (Control Sets of Linear Systems):
Consider the linear control system (4.2). *Assume that the control range U is compact and convex with $0 \in \mathrm{int}\, U$, and that A is hyperbolic and the Kalman rank condition is satisfied. Then the closure Q of the unique control set D is compact and for all compact sets $K \subset D$ with $\mathrm{int}\, K \neq \emptyset$ it holds that*

$$\boxed{h^*_{\mathrm{inv}}(K, Q) = h_{\mathrm{inv}}(K, Q) = \sum_{i:\ \mathrm{Re}(\lambda_i)>0} \mathrm{Re}(\lambda_i),}$$

where $\lambda_1, \ldots, \lambda_d$ are the eigenvalues of A.

Proof:
For every $\rho \in (0,1]$ we consider the system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad u \in \mathcal{U}^\rho = \{u \in \mathcal{U} \mid u(t) \in U^\rho \text{ a.e.}\},$$

where $U^\rho = \rho \cdot U$. That is, we consider the same linear system with smaller control range. It is easy to see that U^ρ is compact and convex with $0 \in \operatorname{int} U$ for all $\rho \in (0,1]$. Hence, for every $\rho \in (0,1]$ there exists a unique control set D^ρ with nonvoid interior and compact closure. By Theorem 4.2.1(i) and Proposition 2.1.13(ii), it follows that

$$h^*_{\mathrm{inv}}(K,Q) = h^*_{\mathrm{inv}}(\tilde{K},Q) \leq h_{\mathrm{inv}}(\tilde{K}),$$

if $\tilde{K} \subset \operatorname{int} Q$ is compact, has nonvoid interior and is controlled invariant. If we can show that $\operatorname{cl} D^\rho \subset \operatorname{int} D$ for some $\rho < 1$, thus we obtain the assertion, since by Theorem 4.1.2 we have

$$\sum_{i:\ \mathrm{Re}(\lambda_i)>0} \mathrm{Re}(\lambda_i) = h_{\mathrm{inv}}(K,Q) \leq h^*_{\mathrm{inv}}(K,Q) \leq h_{\mathrm{inv}}(\operatorname{cl} D^\rho) = \sum_{i:\ \mathrm{Re}(\lambda_i)>0} \mathrm{Re}(\lambda_i).$$

Note that here we use that D^ρ and hence $\operatorname{cl} D^\rho$ is controlled invariant with respect to system (4.2) for every $\rho \in (0,1]$. From Colonius & Spadini [17, Theorem 4.1] it follows that $0 \in \operatorname{int} D^\rho$ for all $\rho \in (0,1]$ and for every neighborhood W of 0 there exists $\rho_0 \in (0,1]$ with $D^\rho \subset W$ for all $\rho \in (0,\rho_0]$. This implies that for small enough ρ we have $\operatorname{cl} D^\rho \subset \operatorname{int} D$. Hence, the assertion holds. □

The following theorem yields a formula for $h^*_{\mathrm{inv}}(K,\operatorname{cl} D)$ for one-dimensional systems with one control vector field and shows that it coincides with $h_{\mathrm{inv}}(K,\operatorname{cl} D)$.

4.2.5 Theorem (Control Sets of One-Dimensional Systems):
On $M = \mathbb{R}$ consider a control-affine system of the form

$$\dot{x}(t) = f_0(x(t)) + u(t) f_1(x(t)), \quad u \in \mathcal{U}. \tag{4.21}$$

Let D be a control set with nonvoid interior and compact closure Q, and assume that local accessibility holds on Q. Then for every compact set $K \subset D$ with nonvoid interior we have

$$\boxed{h^*_{\mathrm{inv}}(K,Q) = h_{\mathrm{inv}}(K,Q) = \max\left\{0, \min_{x\in Q}\left[f_0'(x) - \frac{f_1'(x)}{f_1(x)} f_0(x)\right]\right\}.} \tag{4.22}$$

Proof:
The proof is subdivided into three steps.

Step 1: By Proposition 1.3.6(i) D is connected and thus Q is a compact interval. In order to show that formula (4.22) makes sense, we have to prove that $f_1(x) \neq 0$ for all $x \in Q$: Assume to the contrary that $f_1(x^*) = 0$ for some $x^* \in Q$. From Colonius & Kliemann [16, Theorem 8.1.1, p. 313] it follows that for every $x \in Q$ there exists $u_x \in U$ with $f_0(x) + u_x f_1(x) = 0$. Hence, $f_0(x^*) = 0$, which implies $\varphi(t,x^*,u) = x^*$ for all $t \in \mathbb{R}$ and $u \in \mathcal{U}$ and therefore contradicts the local accessibility on Q.

<u>*Step 2:*</u> We prove that system (4.21) is topologically conjugate to a system with constant control vector field via a C^2-diffeomorphism: Since $f_1(x) \neq 0$ on Q, by continuity of f_1, it holds that $f_1(x) \neq 0$ on a neighborhood of Q. Since Q is a compact interval, this neighborhood can be chosen as an open interval (a, b). On (a, b) there exists a C^2-function π with derivative $f_1(x)^{-1}$, i.e., $\pi(x) = \int f_1(x)^{-1} dx$. Since $f_1(x)$ is either strictly positive or strictly negative on (a, b), π is either strictly increasing or strictly decreasing. Hence, we can extend π to a C^2-diffeomorphism $\pi : \mathbb{R} \to \mathbb{R}$ and define vector fields $g_0, g_1 : \mathbb{R} \to \mathbb{R}$ by

$$g_i(x) := \pi'(\pi^{-1}(x))f_i(\pi^{-1}(x)) \quad \text{for all } x \in \mathbb{R}, \quad i = 0, 1.$$

Then it holds that $g_1(x) \equiv f_1(\pi^{-1}(x))^{-1} f_1(\pi^{-1}(x)) \equiv 1$ and system (4.21) is, by Proposition 2.2.8, topologically conjugate to the system

$$\dot{y}(t) = g_0(y(t)) + u(t), \quad u \in \mathcal{U}, \tag{4.23}$$

with topological conjugacy $(\pi, \mathrm{id}_{\mathcal{U}})$. Moreover, $\pi(D)$ is a bounded control set for system (4.23), since π maps trajectories of (4.21) onto trajectories of (4.23) corresponding to the same control function. By Proposition 2.2.8 it follows that $h^*_{\mathrm{inv}}(K, Q; f_0, f_1)$ equals $h^*_{\mathrm{inv}}(\pi(K), \pi(Q); g_0, g_1)$.

<u>*Step 3:*</u> We compute $h^*_{\mathrm{inv}}(\pi(K), \pi(Q); g_0, g_1)$: By Theorem 3.2.1 we have

$$h^*_{\mathrm{inv}}(K, Q) \geq h_{\mathrm{inv}}(K, Q) \geq \max\left\{0, \min_{y \in \pi(Q)} g_0'(y)\right\}.$$

For the derivative of g_0 we obtain

$$\begin{aligned} g_0'(y) &= f_0'(\pi^{-1}(y)) \frac{\pi'(\pi^{-1}(y))}{\pi'(\pi^{-1}(y))} + \frac{\pi''(\pi^{-1}(x))}{\pi'(\pi^{-1}(y))} f_0(\pi^{-1}(y)) \\ &= f_0'(\pi^{-1}(y)) - \frac{f_1'(\pi^{-1}(y))}{f_1^2(\pi^{-1}(y))} f_1(\pi^{-1}(y)) f_0(\pi^{-1}(y)) \\ &= f_0'(\pi^{-1}(y)) - \frac{f_1'(\pi^{-1}(y))}{f_1(\pi^{-1}(y))} f_0(\pi^{-1}(y)). \end{aligned}$$

Hence,

$$h^*_{\mathrm{inv}}(K, Q) \geq h_{\mathrm{inv}}(K, Q) \geq \max\left\{0, \min_{x \in Q}\left[f_0'(x) - \frac{f_1'(x)}{f_1(x)} f_0(x)\right]\right\}.$$

In order to show the reverse inequality, we distinguish two cases:

Case 1: There exists a point $y^* \in \operatorname{int} \pi(Q)$ with $g_0'(y^*) < 0$: Then, by Remark 4.2.2, $h^*_{\mathrm{inv}}(K, Q) = 0$, since y^* becomes an attractive equilibrium for the constant control function $u(t) \equiv -g_0(y^*)$. Hence, the assertion holds true in this case, since

$$\min_{x \in Q}\left[f_0'(x) - \frac{f_1'(x)}{f_1(x)} f_0(x)\right] = \min_{y \in \pi(Q)} g_0'(y) \leq g_0'(y^*) < 0$$

and thus

$$0 \leq h_{\mathrm{inv}}(K, Q) \leq h^*_{\mathrm{inv}}(K, Q) = 0 = \max\left\{0, \min_{x \in Q}\left[f_0'(x) - \frac{f_1'(x)}{f_1(x)} f_0(x)\right]\right\}.$$

Case 2: For all $y \in \operatorname{int} \pi(Q)$ it holds that $g_0'(y) \geq 0$: Then, by continuity, $g_0'(y) \geq 0$ holds on $\pi(Q)$. By Proposition 2.1.13(ii) we obtain

$$h_{\text{inv}}^*(\pi(K), \pi(Q)) = h_{\text{inv}}^*(\tilde{K}, \pi(Q)) \leq h_{\text{inv}}(\tilde{K})$$

for every compact set $\tilde{K} \subset \operatorname{int} \pi(Q)$ with $\operatorname{int} \tilde{K} \neq \emptyset$ and hence

$$h_{\text{inv}}^*(\pi(K), \pi(Q)) \leq \inf_{\tilde{K}} h_{\text{inv}}(\tilde{K}),$$

where the infimum is taken over all such sets. By Theorem 3.1.4 we obtain

$$h_{\text{inv}}(\tilde{K}) \leq \max\left\{0, \max_{y \in \tilde{K}} g_0'(y)\right\} \cdot \underbrace{\dim_F(\tilde{K})}_{=1} = \max_{y \in \tilde{K}} g_0'(y).$$

Note that $\dim_F(\tilde{K}) = 1$ follows from Proposition A.2.3. Consequently,

$$h_{\text{inv}}^*(\pi(K), \pi(Q)) \leq \inf_{\tilde{K}} \max_{y \in \tilde{K}} g_0'(y) = \min_{y \in Q} g_0'(y) = \min_{x \in Q} \left[f_0'(x) - \frac{f_1'(x)}{f_1(x)} f_0(x) \right],$$

which implies the assertion. □

4.2.6 Remarks:

(i) Theorem 4.2.5 can also be applied to systems on the circle, if the closure of the control set D is not the whole circle.

(ii) Note that $f_0'(x) - \frac{f_1'(x)}{f_1(x)} f_0(x) = D_1 F(x, u_x)$, where $F(x, u) = f_0(x) + u f_1(x)$ and $u_x \in U$ is chosen such that $F(x, u_x) = 0$, i.e., $u_x = -\frac{f_0(x)}{f_1(x)}$. This implies

$$\boxed{h_{\text{inv}}^*(K, Q) = \max\left\{0, \min_{\substack{(x,u) \in Q \times U \\ F(x,u)=0}} D_1 F(x, u)\right\}.} \tag{4.24}$$

(iii) By Colonius & Kliemann [16, Theorem 8.1.2, p. 313] the following holds: If the control set D is contained in a compact positively invariant set and $\operatorname{cl} D$ is a chain control set,[9] then the Lyapunov spectrum on $Q = \operatorname{cl} D$, i.e., the set of all Lyapunov exponents corresponding to solutions, which stay in Q for all positive times, is given by

$$\Sigma_{\text{Ly}}(Q) = \left[\inf_{x \in Q} \max_{\substack{u \in U \\ F(x,u)=0}} D_1 F(x, u), \sup_{x \in Q} \max_{\substack{u \in U \\ F(x,u)=0}} D_1 F(x, u) \right].$$

Since for the system (4.21) there exists exactly one $u \in U$ for each $x \in Q$ such that $F(x, u) = 0$, from Formula (4.24) it follows that

$$\boxed{h_{\text{inv}}^*(K, Q) = \max\{0, \min \Sigma_{\text{Ly}}(Q)\}.}$$

[9]See Colonius & Kliemann [16, Section 3.4] for the definition of chain control sets and their properties.

(iv) Consider a one-dimensional control system of the form

$$\dot{x}(t) = f_0(x(t)) + g(u(t))f_1(x(t)), \quad u \in \mathcal{U}, \tag{4.25}$$

with a homeomorphism $g : \mathbb{R} \to \mathbb{R}$ and compact connected control range $U \subset \mathbb{R}$. This system is topologically conjugate to the control-affine system with right-hand side $F(x,u) = f_0(x) + uf_1(x)$ and control range $g(U)$ via the conjugacy $(\mathrm{id}_{\mathbb{R}}, G)$, where $G(u)(t) := g(u(t))$ for all $u \in \mathcal{U}$ and $t \in \mathbb{R}$, as can easily be verified using Proposition 2.2.8. Hence, by Proposition 2.2.8, it follows that Formula (4.21) also holds for system (4.25).

4.2.7 Example:
Consider a bilinear control system on $\mathbb{R}^2$ of the form

$$\dot{x}(t) = (A_0 + u(t)A_1)\, x(t), \quad u \in \mathcal{U}. \tag{4.26}$$

Let $A_0 = (a^0_{ij})$, $A_1 = (a^1_{ij})$ and $A(u) = A_0 + uA_1$. Consider the projection of (4.26) to S^1, given by (cf. Example 2.2.13)

$$\dot{s}(t) = \big(A(u(t)) - s(t)^T A(u(t))s(t)I\big)\, s(t), \quad u \in \mathcal{U}. \tag{4.27}$$

We assume that local accessibility holds for this system and that $D \subset S^1$ is a control set with nonvoid interior such that $Q := \mathrm{cl}\, D$ is not the whole circle. We want to compute $h^*_{\mathrm{inv}}(K,Q)$ for arbitrary compact $K \subset D$ with $\mathrm{int}\, K \neq \emptyset$. This also gives us the invariance entropy $h^*_{\mathrm{inv,nc}}(\pi^{-1}(K), \pi^{-1}(Q); \pi)$ with respect to system (4.26) in the sense of Definition 2.2.11, where $\pi : \mathbb{R}^2 \backslash \{0\} \to S^1$, $x \mapsto \frac{x}{\|x\|}$. To this end, we describe system (4.27) in polar coordinates. By writing $s(t) = (\cos(\varphi(t)), \sin(\varphi(t)))$ a simple calculation leads to the system

$$\dot{\varphi}(t) = f_0(\varphi(t)) + u(t)f_1(\varphi(t)), \quad u \in \mathcal{U},$$

where $f_0, f_1 : [0, 2\pi) \to \mathbb{R}$ are given by

$$f_k(\varphi) = (a^k_{22} - a^k_{11})\sin(\varphi)\cos(\varphi) - a^k_{12}\sin^2(\varphi) + a^k_{21}\cos^2(\varphi), \quad k = 0, 1.$$

For the derivatives f_k' $(k = 0, 1)$ we get

$$f_k'(\varphi) = (a^k_{22} - a^k_{11})\cos(2\varphi) - (a^k_{12} + a^k_{21})\sin(2\varphi).$$

By Theorem 4.2.5 we obtain that $h^*_{\mathrm{inv}}(K,Q)$ is the maximum of zero and the minimum of the following function on Q.

$$\begin{aligned}\varphi \mapsto\ & (a^0_{22} - a^0_{11})\cos(2\varphi) - (a^0_{12} + a^0_{21})\sin(2\varphi) \\ & - \frac{\big((a^1_{22}-a^1_{11})\cos(2\varphi) - (a^1_{12}+a^1_{21})\sin(2\varphi)\big)\big((a^0_{22}-a^0_{11})\sin(\varphi)\cos(\varphi) - a^0_{12}\sin^2(\varphi) + a^0_{21}\cos^2(\varphi)\big)}{(a^1_{22}-a^1_{11})\sin(\varphi)\cos(\varphi) - a^1_{12}\sin^2(\varphi) + a^1_{21}\cos^2(\varphi)}.\end{aligned}$$

The next example provides an application of this formula. ◊

4.2.8 Example:

We consider the scalar second-order system

$$\ddot{y}(t) + 2b\dot{y}(t) - (1 + u(t))y(t) = 0, \quad u \in \mathcal{U},$$

with $b > 0$ and control range $U = [-\rho, \rho]$, where $0 < \rho < b^2 + 1$. This equation describes the linearization of a controlled damped mathematical pendulum at the unstable position (a linear oscillator). The corresponding first-order system is the following bilinear control system:

$$\dot{x}(t) = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & -2b \end{pmatrix}}_{=:A_0} x(t) + u(t) \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{=:A_1} x(t), \quad u \in \mathcal{U}.$$

The eigenvalues of the matrix A_0 are given by

$$\lambda_{\pm} = -b \pm \sqrt{b^2 + 1}.$$

Since $b > 0$, λ_- is negative and λ_+ is positive. Hence, the uncontrolled system has one stable and one unstable direction. By Example 4.2.7 the projected system on S^1 is given by[10]

$$\dot{\varphi} = \left(-2b\sin(\varphi)\cos(\varphi) - \sin^2(\varphi) + \cos^2(\varphi)\right) + u(t)\cos^2(\varphi), \quad u \in \mathcal{U}.$$

From Colonius & Kliemann [16, Theorem 8.1.1, p. 313] it follows that the control sets on S^1 consist of equilibria. Hence, in order to determine these sets, we have to find the zeros of the right-hand side. To this end, we divide by $\cos^2(\varphi)$ (which is possible for $\varphi \notin \{\frac{\pi}{2}, \frac{3\pi}{2}\}$). This yields

$$\tan^2(\varphi) \;+\; 2b\tan(\varphi) - (1 + u) = 0 \quad \Leftrightarrow \quad \tan(\varphi) = -b \pm \sqrt{b^2 + 1 + u}.$$

Hence, we obtain the solutions

$$\varphi_{\pm} = \arctan\left(-b \pm \sqrt{b^2 + 1 + u}\right) \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)$$

and two other solutions in $(\frac{\pi}{2}, \frac{3\pi}{2})$. The solutions $\varphi_{\pm}$ are real numbers, since

$$b^2 + 1 + u \in [b^2 + 1 - \rho, b^2 + 1 + \rho] \subset (0, 2(b^2 + 1)).$$

Hence, in $(-\frac{\pi}{2}, \frac{\pi}{2})$ we obtain the following two intervals of equilibria, which are the closures of control sets:

$$\begin{aligned} Q_- &= \left[\arctan\left(-b - \sqrt{b^2 + 1 + \rho}\right), \arctan\left(-b - \sqrt{b^2 + 1 - \rho}\right)\right], \\ Q_+ &= \left[\arctan\left(-b + \sqrt{b^2 + 1 - \rho}\right), \arctan\left(-b + \sqrt{b^2 + 1 + \rho}\right)\right]. \end{aligned}$$

Applying the result from Example 4.2.7 we can calculate the invariance entropy of these control sets. An elementary computation gives

$$h_{\text{inv}}(K, Q_{\pm}) = \max\left\{0, \min_{\varphi \in Q_{\pm}} \left(-2b - 2\tan(\varphi)\right)\right\}.$$

[10] The argument t is suppressed from here on.

Hence, we obtain

$$\begin{aligned} h_{\text{inv}}(K, Q_-) &= \max\left\{0, \min_{u\in[-\rho,\rho]}\left(2\sqrt{b^2+1-u}\right)\right\} = 2\sqrt{b^2+1-\rho}, \\ h_{\text{inv}}(K, Q_+) &= 0. \end{aligned}$$

We can interpret this result as follows: The control set $D = \operatorname{int} Q_-$ contains $\varphi_0 := \arctan(-b-\sqrt{b^2+1})$, which is an equilibrium for the control $u=0$, i.e., the vector $(\cos(\varphi_0), \sin(\varphi_0))$ is an eigenvector of the matrix A_0 corresponding to the stable eigenvalue $\lambda_- = -b-\sqrt{b^2+1}$. On D the projected system is controllable. This implies that the cone $\pi^{-1}(D) \subset \mathbb{R}^2$ over D is the maximal subset of $\mathbb{R}^2$ where it is possible to steer to the stable axis (i.e., to the one-dimensional eigenspace corresponding to λ_-) with the bilinear system. Thus, we have computed the invariance entropy of the maximal subset of $\mathbb{R}^2$, where the system can be stabilized to the equilibrium $(0,0)$. The control set Q_+ is easily seen to be invariant and hence its invariance entropy is zero. $\diamondsuit$

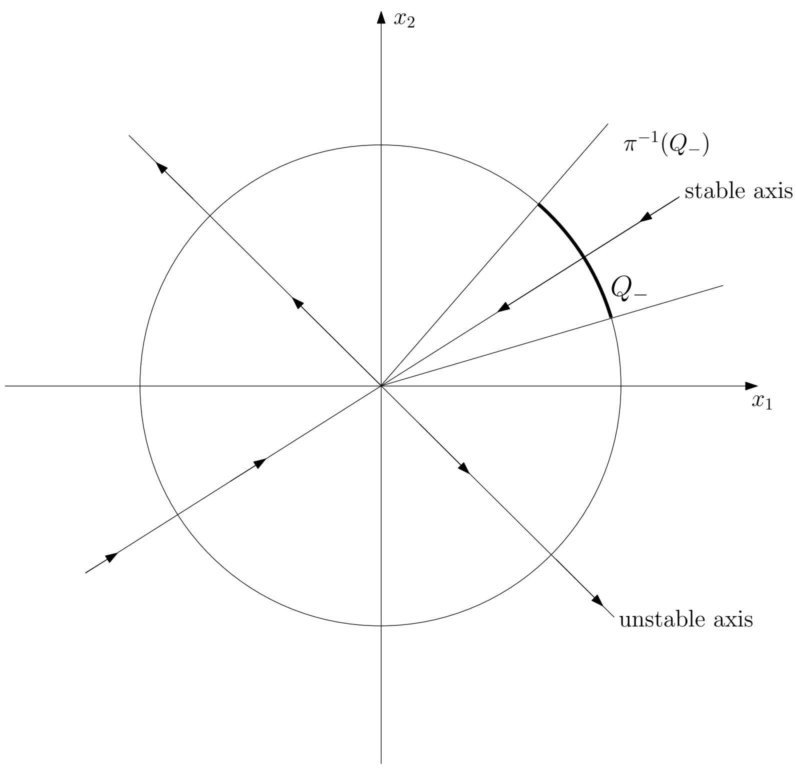

Figure 4.1: The control set Q_-

The next theorem yields an upper bound for the strict invariance entropy $h^*_{\text{inv}}(K, \operatorname{cl} D)$ of a control set D in terms of the positive Lyapunov exponents of a periodic trajectory contained in the interior of D with controllable linearization. The key idea of the proof is as follows: We use the approximate controllability in D in order to steer from every point of the set K into a small neighborhood of a point on the periodic orbit, and then use the controllability of the linearized system in order to keep the system in a small tube around the periodic orbit for some time $\tau > 0$. The number of control functions necessary to do so can be related to the Lyapunov exponents of the linearized system, which then yields an estimate on the strict invariance entropy. Actually, the proof is a modification of that of Theorem 3 in Nair & Evans & Mareels & Moran [42].[11]

4.2.9 Theorem (Upper Bound for the Entropy of a Control Set): *Consider the control-affine system* (1.34) *and let $D \subset M$ be a control set with nonvoid interior and compact closure $Q = \operatorname{cl} D$. Let $(\varphi(\cdot, x_0, u_0), u_0)$ be a T_0-periodic controlled trajectory with $\varphi([0, T_0], x_0, u_0) \subset \operatorname{int} D$. Moreover, assume that $u_0(t)$ is contained in a compact subset of $\operatorname{int} U$ for almost all $t \in [0, T_0]$ and that the linearization along $(\varphi(\cdot, x_0, u_0), u_0)$ is controllable in the sense of Definition 1.2.25. Let $\rho_1, \ldots, \rho_r$ be the different Lyapunov exponents of the solution $\varphi(\cdot, x_0, u_0)$ with corresponding multiplicities $d_1, \ldots, d_r$. Then for every compact set $K \subset D$ it holds that*

$$\boxed{h^*_{\text{inv}}(K, Q) \leq \sum_{j:\ \rho_j > 0} d_j \rho_j.} \tag{4.28}$$

Proof:
We prove the theorem in three steps. In the first step, we use the fundamental lemma of Floquet theory in order to write the solutions of the linearization along $(\varphi(\cdot, x_0, u_0), u_0)$ in terms of the matrix exponential of an endomorphism R on $T_{x_0}M$. Then we construct an adapted Riemannian metric, which yields an orthonormal Jordan basis for R. In the second step, we define several constants. In particular, a (large) time step $\tau \in T_0\mathbb{N}$ and a (small) radius $b_0 > 0$ are defined such that the controllability of the linearization can be used in order to steer the system from the ball $B_{b_0}(x_0)$ to itself in time τ, using a finite number of control functions that is related to the eigenvalues of R and hence to the Lyapunov exponents $\rho_1, \ldots, \rho_r$. This is done in Step 3 by subdividing a cube of side length $2b_0$ centered at the origin of $T_{x_0}M$ into an appropriate number of subcuboids whose midpoints are steered to $0 \in T_{x_0}M$ in time τ via the linearization. Using the Riemannian exponential function at x_0 it is shown that the corresponding control functions also work for the nonlinear system in order to get back to $B_{b_0}(x_0)$ in time τ. This process can be repeated and thus yields $k\tau$-spanning sets for $(B_{b_0}(x_0), Q)$ for all $k \in \mathbb{N}$. By choosing τ big enough and b_0 small enough, the corresponding cardinality growth rate of these sets comes arbitrarily close to $\sum_{j:\ \rho_j > 0} d_j \rho_j$. Since $h^*_{\text{inv}}(K, Q)$ does not depend on the set K, this proves the assertion.

[11] The theorem states that the *local topological feedback entropy* of a continuously differentiable discrete-time system $x_{k+1} = F(x_k, u_k)$ at an equilibrium pair (x_*, u_*) is given by $\sum_{\eta \in \sigma(A):|\eta|>1} \log_2 |\eta|$, where $A = D_x F(x_*, u_*)$ and $B = D_u F(x_*, u_*)$, provided that (A, B) is controllable.

<u>*Step 1:*</u> Consider the automorphism

$$A := D\varphi_{2T_0}(x_0, u_0)(\cdot, 0) \overset{(1.26)}{=} \varphi^l(2T_0, \cdot, 0) : T_{x_0}M \to T_{x_0}M. \tag{4.29}$$

By Proposition 1.2.22(iv) it holds that $A = \varphi^l(T_0, \cdot, 0)^2$ and hence from Chicone [12, Theorem 2.4.7, p. 163][12] it follows that there exists a linear endomorphism $R : T_{x_0}M \to T_{x_0}M$ with

$$A = e^{2T_0 R}. \tag{4.30}$$

By Proposition 1.2.22(iv) it follows that

$$\varphi^l(2T_0 k, \lambda, 0) = A^k \lambda = e^{2T_0 kR}\lambda \text{ for all } \lambda \in T_{x_0}M,\ k \in \mathbb{N}. \tag{4.31}$$

The real parts of the eigenvalues of R are exactly the Lyapunov exponents of the solution $\varphi(\cdot, x_0, u_0)$. To show this, we write $t > 0$ as $t = 2T_0 k + s$ with $k \in \mathbb{N}_0$ and $s \in [0, 2T_0)$. Then for all $\lambda \in T_{x_0}M$ we obtain

$$\varphi^l(t, \lambda, 0) = \varphi^l(s, \varphi^l(k(2T_0), \lambda, 0), 0) \overset{(4.31)}{=} \varphi^l(s, \cdot, 0)e^{2T_0 kR}\lambda.$$

Hence,

$$l_1 \left\| e^{2kT_0 R}\lambda \right\| \le \left\| \varphi^l(t, \lambda, 0) \right\| \le l_2 \left\| e^{2kT_0 R}\lambda \right\|$$

with the positive constants

$$l_1 := \min_{s \in [0, 2T_0]} \left\| \varphi^l(s, \cdot, 0)^{-1} \right\|^{-1}, \quad l_2 := \max_{s \in [0, 2T_0]} \left\| \varphi^l(s, \cdot, 0) \right\|.$$

By Proposition 1.2.22(ii) (or Proposition 1.2.17) we have $D\varphi_{t,u_0}(x_0)\lambda = \varphi^l(t, \lambda, 0)$ and hence the exponential growth of $\|D\varphi_{t,u_0}(x_0)\lambda\|$ for $t \to \infty$ equals the growth of $\|e^{2T_0 \lfloor \frac{t}{2T_0} \rfloor R}\lambda\|$ for all $\lambda \in T_{x_0}M$, which proves the claim.
Choose a basis B_{x_0} of $T_{x_0}M$ adapted to the (real) Jordan structure of R. Let $L_1(R), \ldots, L_r(R)$ be the different Lyapunov spaces of R. Then we have the decomposition

$$T_{x_0}M = L_1(R) \oplus \cdots \oplus L_r(R).$$

Let $d_j = \dim L_j(R)$ and denote by $\lambda^{(j)} \in L_j(R)$ the j^{th} component of a vector $\lambda \in T_{x_0}M$ with respect to this decomposition. Moreover, denote by ρ_j the real part of the eigenvalues corresponding to $L_j(R)$. The restriction of R to $L_j(R)$ is denoted by R_j. Now, let g be a Riemannian metric on M of class C^∞ such that the basis B_{x_0} is orthonormal with respect to g_{x_0}, and let d denote the distance on M induced by g.[13] In order to get a metric with this property one can start with an arbitrary metric $\tilde{g}$ on M, whose existence is guaranteed by Gallot & Hulin & Lafontaine [22, Theorem 2.2, p. 49]. Then one takes a chart (ϕ, V) around x_0 and a scalar product $(\cdot, \cdot)$ on $\mathbb{R}^d$ such that B_{x_0} is orthonormal with

[12] The theorem says that for a nonsingular real $n \times n$-matrix C there exists a real $n \times n$-matrix B with $e^B = C^2$.

[13] Note that by Remark 1.2.23 controllability of the linearization does not depend on the Riemannian metric imposed on M.

respect to the induced scalar product $(D\phi_{x_0}\cdot, D\phi_{x_0}\cdot)$ on $T_{x_0}M$. On V consider the pullback $\hat{g}$ of $(\cdot,\cdot)$ by ϕ, i.e.,

$$\hat{g}(x)(v,w) := (D\phi_x v, D\phi_x w) \text{ for all } x \in V,\ v,w \in T_xM.$$

Then let $\theta : M \to [0,1]$ be a cut-off function of class C^∞ such that $\operatorname{supp}\theta \subset V$ and $\theta(x) = 1$ on a compact neighborhood W of x_0 (see Lemma A.3.3). Define g by

$$g(x) := \begin{cases} \theta(x)\hat{g}(x) + (1-\theta(x))\tilde{g}(x) & \text{for all } x \in V, \\ \tilde{g}(x) & \text{for all } x \in M \backslash V. \end{cases}$$

It can easily be seen that g is a Riemannian metric on M with g_{x_0} having the desired property.

<u>*Step 2:*</u> We fix some constants: Let S_0 be any real number which satisfies

$$S_0 > \sum_{j:\ \rho_j > 0} d_j \rho_j. \tag{4.32}$$

Choose $\xi = \xi(S_0) > 0$ such that

$$0 < d\xi < S_0 - \sum_{j:\ \rho_j > 0} d_j \rho_j. \tag{4.33}$$

Let $\delta \in (0,\xi)$ be chosen small enough such that $\rho_j < 0$ implies $\rho_j + \delta < 0$ for all $j \in \{1,\dots,r\}$. From Lemma A.3.8 it follows that there exists a constant $c = c(\delta) > 0$ such that

$$\forall j \in \{1,\dots,r\} : \ \forall k \in \mathbb{N}_0 : \ \left\| e^{kT_0 R_j} \right\| \le c e^{(\rho_j+\delta)kT_0}, \tag{4.34}$$

where $\|\cdot\|$ denotes the operator norm on $\operatorname{Hom}(T_{x_0}M, T_{x_0}M)$ induced by g_{x_0}. For every $t > 0$ we define positive integers

$$M_j(t) := \begin{cases} \lfloor e^{(\rho_j+\xi)t} \rfloor + 1 & \text{if } \rho_j \ge 0 \\ 1 & \text{if } \rho_j < 0 \end{cases}, \quad j = 1,\dots,r. \tag{4.35}$$

Moreover, we define a function $\beta : (0,\infty) \to (0,\infty)$ by

$$\beta(t) := c\sqrt{r} \max_{1 \le j \le r} \left[e^{(\rho_j+\delta)t} \frac{\sqrt{d_j}}{M_j(t)} \right]. \tag{4.36}$$

If $\rho_j < 0$, then (by our definitions) $\rho_j + \delta < 0$ and $M_j(t) \equiv 1$. This implies that $e^{(\rho_j+\delta)t}\frac{\sqrt{d_j}}{M_j(t)}$ converges to zero for $t \to \infty$. If $\rho_j \ge 0$, we have $M_j(t) \ge e^{(\rho_j+\xi)t}$ by (4.35) and hence

$$e^{(\rho_j+\delta)t} \frac{\sqrt{d_j}}{M_j(t)} \le e^{(\rho_j+\delta)t} \frac{\sqrt{d_j}}{e^{(\rho_j+\xi)t}} = \sqrt{d_j} e^{(\delta-\xi)t}.$$

Since $\delta \in (0,\xi)$, we have $\delta - \xi < 0$ and hence the term above converges to zero for $t \to \infty$. Thus, also $\beta(t) \to 0$ for $t \to \infty$. This implies that for given $\varepsilon > 0$ we can choose a number $\tau = 2kT_0$ with $k \in \mathbb{N}$ big enough such that

$$\beta(\tau) < 1 \ \text{ and } \ \tfrac{d}{\tau}\ln(2) < \varepsilon. \tag{4.37}$$

Since we assume that the linearization along $(\varphi(\cdot, x_0, u_0), u_0)$ is controllable, by Proposition 1.2.27 there exists a constant $C > 0$ with the following property:[14]

$$\forall \lambda \in T_{x_0}M : \ \exists \mu \in L^\infty([0,\tau], \mathbb{R}^m) : \ \varphi^l(\tau, \lambda, \mu) = 0 \text{ and } \|\mu\|_{[0,\tau]} \leq C\|\lambda\|. \tag{4.38}$$

Let $W_1 \subset T_{x_0}M$ and $W_2 \subset M$ be open neighborhoods of $0 \in T_{x_0}M$ and x_0, respectively, such that

$$\exp_{x_0} : W_1 \to W_2$$

is a diffeomorphism. The inverse of $\exp_{x_0}|_{W_1}$ will simply be denoted by $\exp_{x_0}^{-1}$. Now, choose $b_0 > 0$ small enough such that the following conditions are satisfied:

$$\left\{\begin{array}{l} \operatorname{cl} B_{b_0}(0) \subset W_1, \\ \operatorname{cl} B_{b_0}(x_0) \subset D, \\ \operatorname{cl} B_{C\sqrt{d}b_0}(u_0(t)) \subset U \quad \text{for almost all } t \in [0, T_0], \\ \varphi(\tau, \operatorname{cl} B_{b_0}(x_0), u) \subset W_2 \quad \text{if } \|u - u_0\|_{[0,\tau]} \leq C\sqrt{d}b_0. \end{array}\right\} \tag{4.39}$$

The second and third inclusion are possible, since $x_0 \in \operatorname{int} D$ and $u_0(t)$ is contained in a compact subset of $\operatorname{int} U$ for almost all $t \in [0, T_0]$. The last one is possible by continuity of $(x, u) \mapsto \varphi(\tau, x, u)$ (see Proposition 1.2.22(i)). By Proposition 1.2.24 there exists a function $\zeta = \zeta_{\tau,\sqrt{d}C} : [0, \alpha) \to \mathbb{R}_0^+$ $(\alpha > 0)$ with

$$\left\|\exp_{x_0}^{-1}(\varphi(\tau, x, u)) - \varphi^l(\tau, \exp_{x_0}^{-1}(x), u - u_0)\right\| \leq \zeta(b)b \tag{4.40}$$

for all $(x, u) \in M \times \mathcal{U}$ with $d(x, x_0) \leq b \leq b_0$ and $\|u - u_0\|_{[0,\tau]} \leq C\sqrt{d}b$, and $\zeta(b) \to 0$ for $b \to 0$. We can assume that $b_0 < \alpha$ and hence $\zeta(b_0)$ is defined. Because of the strict inequality $\beta(\tau) < 1$ we can also assume that b_0 is chosen small enough such that

$$\sqrt{r}\zeta(b_0) + \beta(\tau) \leq 1. \tag{4.41}$$

<u>*Step 3:*</u> By Theorem 4.2.1(i) and (4.39), we can assume that $K = \operatorname{cl} B_{b_0}(x_0)$. Consider a d-dimensional cube $\mathcal{C}$ in $T_{x_0}M$ centered at the origin with sides of length $2b_0$ parallel to the vectors of the basis B_{x_0}. Then $\exp_{x_0}^{-1}(K) = \operatorname{cl} B_{b_0}(0) \subset T_{x_0}M$ and hence $\exp_{x_0}^{-1}(K) \subset \mathcal{C}$. Partition $\mathcal{C}$ by dividing each coordinate axis corresponding to a component of the j^{th} Lyapunov space of R into $M_j(\tau)$ intervals of equal length. The total number of subcuboids in this partition is $\prod_{j=1}^r M_j(\tau)^{d_j}$. Now pick an arbitrary $x \in \operatorname{cl} B_{b_0}(x_0)$. Let $\gamma_0 : [0,1] \to M$ be a shortest geodesic from x_0 to x and let $\lambda_x \in \mathcal{C}$ be the center of a subcuboid which contains $\exp_{x_0}^{-1}(x) = \dot\gamma_0(0)$. (Note that $\|\dot\gamma_0(0)\| = \mathcal{L}(\gamma_0) = d(x_0, x) \leq b_0$.) Then the following estimate holds, where the additional superscripts denote components of vectors within the corresponding Lyapunov spaces of R:

$$\begin{aligned} \left\|\dot\gamma_0(0)^{(j)} - \lambda_x^{(j)}\right\| &= \left[\sum_{l=1}^{d_j} \left(\dot\gamma_0(0)^{(j,l)} - \lambda_x^{(j,l)}\right)^2\right]^{1/2} \\ &\leq \left[\sum_{l=1}^{d_j} \left(\frac{b_0}{M_j(\tau)}\right)^2\right]^{1/2} = \frac{\sqrt{d_j}}{M_j(\tau)} b_0. \end{aligned} \tag{4.42}$$

[14]Note that controllability on $[0, T_0]$ implies controllability on $[0, \tau]$.

By (4.38) there exists $u_x \in L^\infty([0,\tau],\mathbb{R}^m)$ such that $\varphi^l(\tau,\lambda_x,u_x-u_0)=0$ or equivalently,

$$\varphi^l(\tau,\lambda_x,u_x)=\varphi^l(\tau,0,u_0) \tag{4.43}$$

and

$$\|u_x-u_0\|_{[0,\tau]} \le C\|\lambda_x\| \le C\left[\sum_{j=1}^{r}\sum_{l=1}^{d_j}\left\|\lambda_x^{(j,l)}\right\|^2\right]^{1/2} \le C\sqrt{d}b_0, \tag{4.44}$$

since $\lambda_x \in \mathcal{C}$ implies $\|\lambda_x^{(j,l)}\| \le b_0$ for each component. By (4.39) it holds that $u_x \in \mathcal{U}$ and

$$\varphi(\tau,x,u_x)\in W_2. \tag{4.45}$$

Let $\gamma_1:[0,1]\to M$ be a shortest geodesic from x_0 to $\varphi(\tau,x,u_x)$. Then

$$d\left(\varphi(\tau,x,u_x),x_0\right)=\mathcal{L}(\gamma_1)=\int_0^1 \underbrace{\|\dot\gamma_1(t)\|}_{=\text{ constant}}\,dt=\|\dot\gamma_1(0)\|\,. \tag{4.46}$$

By the triangle inequality we have

$$\left\|\dot\gamma_1(0)^{(j)}\right\| \le \left\|\dot\gamma_1(0)^{(j)}-\varphi^l\left(\tau,\dot\gamma_0(0),u_x-u_0\right)^{(j)}\right\|+\left\|\varphi^l(\tau,\dot\gamma_0(0),u_x-u_0)^{(j)}\right\|.$$

Since g is chosen such that the Lyapunov spaces of R are orthogonal, for the first term we obtain

$$\begin{aligned}\left\|\dot\gamma_1(0)^{(j)}\right. &\left.-\ \varphi^l(\tau,\dot\gamma_0(0),u_x-u_0)^{(j)}\right\| = \left\|\left[\dot\gamma_1(0)-\varphi^l(\tau,\dot\gamma_0(0),u_x-u_0)\right]^{(j)}\right\|\\ &\le \left\|\dot\gamma_1(0)-\varphi^l(\tau,\dot\gamma_0(0),u_x-u_0)\right\|\\ &= \left\|\exp_{x_0}^{-1}(\varphi(\tau,x,u_x))-\varphi^l(\tau,\exp_{x_0}^{-1}(x),u_x-u_0)\right\| \overset{(4.40)}{\le} \zeta(b_0)b_0.\end{aligned}$$

By linearity of $\varphi^l(\tau,\cdot,\cdot)$ for the second term we obtain

$$\begin{aligned}\left\|\varphi^l(\tau,\dot\gamma_0(0),u_x-u_0)^{(j)}\right\| &= \left\|\varphi^l(\tau,\dot\gamma_0(0),u_x)^{(j)}-\varphi^l(\tau,0,u_0)^{(j)}\right\|\\ &\overset{(4.43)}{=} \left\|\varphi^l(\tau,\dot\gamma_0(0),u_x)^{(j)}-\varphi^l(\tau,\lambda_x,u_x)^{(j)}\right\|\\ &= \left\|\varphi^l(\tau,\dot\gamma_0(0)-\lambda_x,0)^{(j)}\right\|\\ &\overset{(4.31)}{=} \left\|\left[e^{2kT_0R}(\dot\gamma_0(0)-\lambda_x)\right]^{(j)}\right\|\\ &= \left\|\left[e^{\tau R}(\dot\gamma_0(0)-\lambda_x)\right]^{(j)}\right\|.\end{aligned}$$

By invariance of the Lyapunov spaces of R under $e^{\tau R}$ we get

$$\begin{aligned}\left\|\varphi^l(\tau,\dot\gamma_0(0),u_x-u_0)^{(j)}\right\| &= \left\|e^{\tau R}(\dot\gamma_0(0)-\lambda_x)^{(j)}\right\|\\ &\le \left\|e^{\tau R_j}\right\|\left\|(\dot\gamma_0(0)-\lambda_x)^{(j)}\right\|\end{aligned}$$

$$\overset{(4.34)}{\leq} ce^{(\rho_j+\delta)\tau}\left\|(\dot{\gamma}_0(0)-\lambda_x)^{(j)}\right\|.$$

Altogether, we have

$$\begin{aligned}\left\|\dot{\gamma}_1(0)^{(j)}\right\| &\leq \zeta(b_0)b_0 + ce^{(\rho_j+\delta)\tau}\left\|(\dot{\gamma}_0(0)-\lambda_x)^{(j)}\right\| \\ &\overset{(4.42)}{\leq} \zeta(b_0)b_0 + ce^{(\rho_j+\delta)\tau}\frac{\sqrt{d_j}}{M_j(\tau)}b_0.\end{aligned}$$

By orthogonality of the Lyapunov spaces of R it follows that

$$\begin{aligned}d\left(\varphi(\tau,x,u_x),x_0\right) &= \|\dot{\gamma}_1(0)\| = \left(\sum_{j=1}^r \left\|\dot{\gamma}_1(0)^{(j)}\right\|^2\right)^{1/2} \\ &\leq \left(\sum_{j=1}^r \left(\zeta(b_0)b_0 + ce^{(\rho_j+\delta)\tau}\frac{\sqrt{d_j}}{M_j(\tau)}b_0\right)^2\right)^{1/2} \\ &\overset{(\Delta)}{\leq} \sqrt{r}\zeta(b_0)b_0 + \left(\sum_{j=1}^r \left(ce^{(\rho_j+\delta)\tau}\frac{\sqrt{d_j}}{M_j(\tau)}b_0\right)^2\right)^{1/2} \\ &\leq \sqrt{r}\zeta(b_0)b_0 + c\sqrt{r}\max_{1\leq j\leq r}\left[e^{(\rho_j+\delta)\tau}\frac{\sqrt{d_j}}{M_j(\tau)}\right]b_0 \\ &\overset{(4.36)}{=} \left[\sqrt{r}\zeta(b_0)+\beta(\tau)\right]b_0 \overset{(4.41)}{\leq} b_0.\end{aligned}$$

The estimate (Δ) follows from the triangle inequality in $\mathbb{R}^r$. Hence, we have proved that $\prod_{j=1}^r M_j(\tau)^{d_j}$ admissible control functions are sufficient to steer all points of K back to K in time τ. By the no-return property of control sets it follows that the trajectories do not leave Q within the time interval $(0,\tau)$. By iterated concatenation of these control functions we can construct an $n\tau$-spanning set for each $n \in \mathbb{N}$ with $(\prod_{j=1}^r M_j(\tau)^{d_j})^n$ elements and hence we obtain

$$r^*_{\text{inv}}(n\tau,K,Q) \leq \left(\prod_{j=1}^r M_j(\tau)^{d_j}\right)^n = \left(\prod_{j:\ \rho_j\geq 0}\left(\lfloor e^{(\rho_j+\xi)\tau}\rfloor+1\right)^{d_j}\right)^n,$$

which implies

$$\begin{aligned}h^*_{\text{inv}}(K,Q) &= \limsup_{n\to\infty}\frac{1}{n\tau}\ln r^*_{\text{inv}}(n\tau,K,Q) \leq \frac{1}{\tau}\sum_{j:\ \rho_j\geq 0}\ln\left(\lfloor e^{(\rho_j+\xi)\tau}\rfloor+1\right)^{d_j} \\ &= \sum_{j:\ \rho_j\geq 0} d_j\frac{1}{\tau}\ln\left(\lfloor e^{(\rho_j+\xi)\tau}\rfloor+1\right) \leq \sum_{j:\ \rho_j\geq 0} d_j\frac{1}{\tau}\ln\left(2e^{(\rho_j+\xi)\tau}\right) \\ &= \sum_{j:\ \rho_j\geq 0} d_j\left(\tfrac{\ln(2)}{\tau}+(\rho_j+\xi)\right) \leq \tfrac{d}{\tau}\ln(2)+d\xi+\sum_{j:\ \rho_j>0} d_j\rho_j \\ &\overset{(4.33)}{<} \tfrac{d}{\tau}\ln(2)+S_0 \overset{(4.37)}{<} S_0+\varepsilon.\end{aligned}$$

The first equality follows from Proposition 2.2.10. Since ε can be chosen arbitrarily small and S_0 arbitrarily close to $\sum_{j:\ \rho_j>0} d_j\rho_j$, the assertion of the theorem follows. $\square$

4.2.10 Remark:
If D is a control set with nonvoid interior for the control-affine system (1.34), and if every point in the interior of D is locally accessible, then by Colonius & Kliemann [16, Proposition 4.3.3, p. 100] the periodic points of the control flow are dense in the lift $\mathcal{D}$ of D. This implies the existence of periodic solutions as required in Theorem 4.2.9. But in general it is not clear if the linearizations along those solutions are controllable.

Since equilibria are periodic solutions for every period $T_0 > 0$, the following corollary immediately follows.

4.2.11 Corollary:
Consider the control-affine system (1.34) *and let g be a Riemannian metric on M of class C^∞. Let $D \subset M$ be a control set with nonvoid interior and compact closure $Q = \mathrm{cl}\, D$. Let $(x_0, u_0) \in \mathrm{int}\, D \times \mathrm{int}\, U$ such that $F(x_0, u_0) = 0$. Moreover, assume that the linearization along (x_0, u_0) is controllable. Then for every compact set $K \subset D$ it holds that*

$$\boxed{h^*_{\mathrm{inv}}(K, Q) \leq \sum_{\substack{\lambda\in\sigma(\nabla F_{u_0}(x_0))\\ \mathrm{Re}(\lambda)>0}} \mathrm{Re}(\lambda),}$$

where every eigenvalue λ in the sum is counted with its multiplicity.

In the proof of the next corollary, we use the fact that in dimension one local accessibility implies controllability of the linearized system at any controlled equilibrium.

4.2.12 Corollary:
Consider the control-affine system (1.34) *on $M = \mathbb{R}$. Let D be a control set with nonvoid interior and compact closure $Q = \mathrm{cl}\, D$ and assume that local accessibility holds on $\mathrm{int}\, D$. Then for every compact set $K \subset D$ it holds that*

$$\boxed{h^*_{\mathrm{inv}}(K, \mathrm{cl}\, D) \leq \max\left\{0, \inf_{\substack{(x,u)\in \mathrm{int}\, D\times \mathrm{int}\, U\\ F(x,u)=0}} D_1F(x,u)\right\}.}$$

Proof:
Let $(x_*, u_*) \in \mathrm{int}\, D \times \mathrm{int}\, U$ such that

$$F(x_*, u_*) = f_0(x_*) + \sum_{i=1}^m u_{*i} f_i(x_*) = 0. \tag{4.47}$$

Then the controllability of the linearization at (x_*, u_*) is equivalent to the controllability of the matrix pair (A, B) with

$$\begin{aligned} A &= D_1F(x_*, u_*) = f_0'(x_*) + \sum_{i=1}^{m} u_{*i} f_i'(x_*) \in \mathbb{R}^{1\times 1}, \\ B &= D_2F(x_*, u_*) = (f_1(x_*), \dots, f_m(x_*)) \in \mathbb{R}^{1\times m}, \end{aligned}$$

which in this case means that the matrix B has full rank (cf. Formula (4.20)). Since B is a $1 \times m$-matrix, this is satisfied if and only if $B \neq 0$. Assume to the contrary that $B = (f_1(x_*), \dots, f_m(x_*)) = 0$. Then from (4.47) it follows that $f_0(x_*) = 0$ and hence x_* is an equilibrium for every control function, which contradicts the local accessibility on Q. Thus, Theorem 4.2.9 implies

$$\begin{aligned} h^*_{\mathrm{inv}}(K, \operatorname{cl} D) &\leq \inf_{\substack{(x,u)\in \operatorname{int} D\times \operatorname{int} U \\ F(x,u)=0}} \max\{0, D_1F(x,u)\} \\ &= \max\left\{0, \inf_{\substack{(x,u)\in \operatorname{int} D\times \operatorname{int} U \\ F(x,u)=0}} D_1F(x,u)\right\} \end{aligned}$$

for every compact set $K \subset D$. □

4.2.13 Corollary:
Let Q be the closure of a relatively compact inner control set of the control-affine system (1.34). Further assume that there exists a T_0-periodic controlled trajectory $(\varphi(\cdot, x_0, u_0), u_0)$ with $\varphi([0,T_0], x_0, u_0) \subset Q$ and $u_0(t) \in U_1$ for almost all $t \in [0, T_0]$. Then, if the linearization along $(\varphi(\cdot, x_0, u_0), u_0)$ is controllable, the estimate

$$\boxed{h_{\mathrm{inv}}(Q) \leq \sum_{i:\ \lambda_i>0} d_i\lambda_i}$$

holds, where $\lambda_1, \dots, \lambda_r$ are the Lyapunov exponents of the solution $\varphi(\cdot, x_0, u_0)$ with corresponding multiplicities $d_1, \dots, d_r$.

Proof:
From Theorem 4.2.9 it follows that

$$h^*_{\mathrm{inv}}(Q, \operatorname{cl} D_\rho; U_\rho) \leq \sum_{i:\ \lambda_i>0} d_i\lambda_i \quad \text{for all } \rho \in [0,1).$$

Now, for given $\varepsilon > 0$ choose $\rho \in [0,1)$ such that $\operatorname{cl} D_\rho \subset N_\varepsilon(Q)$. Then

$$h_{\mathrm{inv}}(\varepsilon, Q; U_0) \leq h_{\mathrm{inv}}(\varepsilon, Q; U_\rho) \leq h^*_{\mathrm{inv}}(Q, \operatorname{cl} D_\rho; U_\rho) \leq \sum_{i:\ \lambda_i>0} d_i\lambda_i.$$

The first inequality follows from $U_\rho \subset U_0$ and Proposition 2.1.8(v). The second is clear, since every T-spanning set for $(Q, \operatorname{cl} D_\rho)$ is also (T, ε)-spanning for Q. (Note that Q is controlled invariant with respect to all the control ranges U_ρ, $\rho \in [0,1]$.) Since $h_{\mathrm{inv}}(Q) = \lim_{\varepsilon \searrow 0} h_{\mathrm{inv}}(\varepsilon, Q; U_0)$, the assertion follows. □

4.2.14 Example:
Consider the bilinear control system (2.18) on $\mathbb{R}^d$ and its projection to the unit sphere S^{d-1}. Let $D \subset S^{d-1}$ be a control set of the projected system with nonvoid interior[15] and set $Q := \operatorname{cl} D$. Since S^{d-1} is compact, also Q is compact. By (3.18) the covariant derivative of the right-hand side vector field F_u of the projected system is

$$\nabla F_u(s) = Q_s(A(u) - s^T A(u) s I)$$

with $Q_s = I - ss^T$. A point $s_0 \in \operatorname{int} D$ is an equilibrium for some constant control $u_0 \in \operatorname{int} U$ if and only if

$$0 = F(s_0, u_0) = A(u_0)s_0 - (s_0^T A(u_0) s_0) s_0,$$

i.e., if and only if s_0 is an eigenvector of $A(u_0)$. We write λ for the corresponding eigenvalue $s_0^T A(u_0) s_0$. We want to compute the spectrum of $\nabla F_{u_0}(s_0)$. To this end, let $v \in T_{s_0} S^{d-1} = s_0^{\perp}$. Then

$$\nabla F_{u_0}(s_0) v \;=\; Q_{s_0}\left(A(u_0)v - (s_0^T A(u_0) s_0) v\right) = Q_{s_0} A(u_0) v - \lambda v.$$

Hence, it suffices to determine the eigenvalues of the linear map $L := Q_{s_0} A(u_0)|_{T_{s_0} S^{d-1}}$. To this end, let $\mu \in \mathbb{C}$ be an arbitrary eigenvalue of $A(u_0)$ and $z \notin \langle s_0 \rangle$ a corresponding (complex) eigenvector. Consider the vector $\zeta := Q_{s_0} z \in T_{s_0} S^{d-1} \oplus i T_{s_0} S^{d-1}$. Then we obtain

$$\begin{aligned} L\zeta \;&=\; Q_{s_0} A(u_0) Q_{s_0} z = Q_{s_0} A(u_0)(I - s_0 s_0^T) z = Q_{s_0} A(u_0) z - Q_{s_0} A(u_0) s_0 s_0^T z \\ &=\; \mu Q_{s_0} z - Q_{s_0}(A(u_0) s_0)(s_0^T z) = \mu\zeta - Q_{s_0}(\lambda s_0)(s_0^T z) \\ &=\; \mu\zeta - \lambda(s_0^T z) Q_{s_0} s_0 = \mu\zeta. \end{aligned}$$

Hence, the eigenvalues of L coincide with eigenvalues of $A(u_0)$ and the eigenvalues of $\nabla F_{u_0}(s_0)$ are the eigenvalues of $A(u_0)$ minus λ. Hence, under the assumption that the linearization at (s_0, u_0) is controllable, by Corollary 4.2.11 the estimate

$$\boxed{h^*_{\mathrm{inv}}(K, Q) \leq \sum_{\substack{j \in \{1,\ldots,d-1\}, \\ \mathrm{Re}(\mu_j) > \lambda}} (\mathrm{Re}(\mu_j) - \lambda)}$$

holds true for every compact set $K \subset D$, where $\lambda, \mu_1, \ldots, \mu_{d-1}$ are the eigenvalues of $A(u_0)$. $\Diamond$

4.2.15 Open Questions:

- Is $h_{\mathrm{inv}}(K, Q) = h^*_{\mathrm{inv}}(K, Q)$ true in general, if Q is the closure of a control set and $K \subset D$ with $\operatorname{int}(K) \neq \emptyset$?
- Does $h^*_{\mathrm{inv}}(K, Q) = h^*_{\mathrm{inv}}(Q)$ and/or $h_{\mathrm{inv}}(K, Q) = h_{\mathrm{inv}}(Q)$ hold for $Q = \operatorname{cl} D$?
- Can the periodic trajectory in Theorem 4.2.9 be replaced by an arbitrary trajectory?

[15] By Colonius & Kliemann [16, Theorem 7.3.3, pp. 283–284] for the projected system on $\mathbb{P}^{d-1}$ there exist (finitely many) control sets with nonvoid interior under the assumption of local accessibility. The connected components of the lifts of these control sets to the unit sphere are control sets for the system on the sphere.

Chapter 5

Alternative Characterization, Data Rates and Numerics

In this last chapter, we provide an alternative characterization of the strict invariance entropy $h^*_{\mathrm{inv}}(Q)$ via so-called invariant coverings of the set Q, which reveals the similarity of the invariance entropy and the topological feedback entropy, defined in Nair & Evans & Mareels & Moran [42]. We use this characterization in order to compute the strict invariance entropy $h^*_{\mathrm{inv}}(Q)$ for one-dimensional linear systems, when Q is a compact interval, and thereby show that it coincides with $h_{\mathrm{inv}}(Q)$ in this case. Moreover, we use it to prove that $h^*_{\mathrm{inv}}(Q)$ equals the infimum data rate in a feedback loop necessary to render the set Q invariant for a very general class of coding and control schemes, in analogy to the corresponding result for the topological feedback entropy proved in [42]. Finally, we use the characterization of $h^*_{\mathrm{inv}}(Q)$ via invariant coverings in order to describe an algorithm for the numerical computation of rigorous upper bounds for $h^*_{\mathrm{inv}}(Q)$, which is mainly based on another algorithm, designed for the computation of topological entropy (see Froyland & Junge & Ochs [21]).

5.1 Characterization via Invariant Coverings

In this section, we give a different characterization of the strict invariance entropy in terms of invariant coverings, which are defined as follows.

5.1.1 Definition (Invariant Covering):
Let Q be a compact controlled invariant set for the control system (1.7). *An* **invariant covering** *of Q is given by a triple $(\mathcal{A}, v, \tau)$, where $\mathcal{A}$ is a finite covering of Q, $v : \mathcal{A} \to \mathcal{U}$ is a function, assigning to each set in $\mathcal{A}$ a control function, and τ is a positive real number such that*

$$\varphi([0,\tau], A, v(A)) \subset Q \quad \text{for all } A \in \mathcal{A}.$$

We also say that the triple $(\mathcal{A}, v, \tau)$ is **invariantly covering** *the set Q.*

The following proposition yields a first relation between the strict invariance entropy and invariant coverings.

5.1.2 Proposition:
Let Q be a compact controlled invariant set for control system (1.7). Then $h^*_{\text{inv}}(Q) < \infty$ if and only if there exists an invariant covering $(\mathcal{A}, v, \tau)$ of Q.

Proof:
Assume that $h^*_{\text{inv}}(Q) < \infty$. Then, by Corollary 2.1.11, there exists a time $\tau > 0$ and a finite τ-spanning set $\mathcal{S}^* = \{v_1, \ldots, v_n\}$ for Q. Define

$$A_j := \{x \in Q \mid \varphi([0,\tau], x, v_j) \subset Q\}, \quad j = 1, \ldots, n.$$

Let $\mathcal{A} := \{A_1, \ldots, A_n\}$ and let $v : \mathcal{A} \to \mathcal{U}$ be given by $v(A_j) := v_j$. Then obviously $(\mathcal{A}, v, \tau)$ is an invariant covering of Q.
On the other hand, if $(\mathcal{A}, v, \tau)$ is an invariant covering of Q, the set $v(\mathcal{A}) \subset \mathcal{U}$ is a finite τ-spanning set for Q. Hence, by Corollary 2.1.11, $h^*_{\text{inv}}(Q) < \infty$. □

5.1.3 Definition (Entropy of an Invariant Covering):
Let $\mathcal{C} = (\mathcal{A}, v, \tau)$ be an invariant covering of a compact controlled invariant set Q with $\mathcal{A} = \{A_1, \ldots, A_q\}$ and let $K \subset Q$ be compact. We denote by v_a the control function $v(A_a)$ for $a = 1, \ldots, q$, and we define for every word $[a_0, a_1, \ldots, a_{N-1}]$ ($N \in \mathbb{N}$ arbitrary) with $a_j \in \{1, \ldots, q\}$ a control function by

$$v_{a_0,a_1,\ldots,a_{N-1}}(t) := v_{a_j}(t - j\tau) \quad \text{for all } t \in [j\tau, (j+1)\tau), \quad j = 0, 1, \ldots, N-1.$$

On $\mathbb{R}\backslash[0, N\tau)$ the function may be extended arbitrarily. The word $[a_0, a_1, \ldots, a_{N-1}]$ is called **admissible** *for the pair (K, Q) and the invariant covering $\mathcal{C}$, if there exists an $x \in K$ with*

$$\varphi\left(j\tau, x, v_{a_0,a_1,\ldots,a_{N-1}}\right) \in A_{a_j} \quad \text{for } j = 0, 1, \ldots, N-1.$$

We write $\mathcal{W}_N(\mathcal{C}; K, Q)$ for the set of all admissible words of length N ($N \in \mathbb{N}$ arbitrary). The **strict invariance entropy of (K, Q) with respect to the invariant covering $\mathcal{C}$** *is defined by*

$$h^*_{\text{inv}}(\mathcal{C}; K, Q) := \limsup_{N \to \infty} \frac{\ln \#\mathcal{W}_N(\mathcal{C}; K, Q)}{N\tau}. \tag{5.1}$$

If $K = Q$, we also write $\mathcal{W}_N(\mathcal{C}; Q)$ and $h^*_{\text{inv}}(\mathcal{C}; Q)$ instead of $\mathcal{W}_N(\mathcal{C}; Q, Q)$ and $h^*_{\text{inv}}(\mathcal{C}; Q, Q)$, respectively.

5.1.4 Proposition:
Let Q be a compact controlled invariant set for control system (1.7). If $\mathcal{C}$ is an invariant covering of Q, then the sequence $(\ln \#\mathcal{W}_N(\mathcal{C}; Q))_{N \in \mathbb{N}}$ is subadditive and hence

$$\boxed{h^*_{\text{inv}}(\mathcal{C}; Q) = \lim_{N \to \infty} \frac{\ln \#\mathcal{W}_N(\mathcal{C}; Q)}{N\tau} = \inf_{N \in \mathbb{N}} \frac{\ln \#\mathcal{W}_N(\mathcal{C}; Q)}{N\tau}.}$$

Proof:
It suffices to show that

$$\#\mathcal{W}_{N_1+N_2}(\mathcal{C};Q) \leq \#\mathcal{W}_{N_1}(\mathcal{C};Q) \cdot \#\mathcal{W}_{N_2}(\mathcal{C};Q) \quad \text{for all } N_1, N_2 \in \mathbb{N}.$$

To this end, we define an injective mapping

$$\alpha : \mathcal{W}_{N_1+N_2}(\mathcal{C};Q) \to \mathcal{W}_{N_1}(\mathcal{C};Q) \times \mathcal{W}_{N_2}(\mathcal{C};Q).$$

This implies

$$\begin{aligned}\#\mathcal{W}_{N_1+N_2}(\mathcal{C};Q) &= \#\alpha\left(\mathcal{W}_{N_1+N_2}(\mathcal{C};Q)\right) \\ &\leq \#\left(\mathcal{W}_{N_1}(\mathcal{C};Q) \times \mathcal{W}_{N_2}(\mathcal{C};Q)\right) = \#\mathcal{W}_{N_1}(\mathcal{C};Q) \cdot \#\mathcal{W}_{N_2}(\mathcal{C};Q).\end{aligned}$$

Let $\mathcal{C} = (\mathcal{A}, v, \tau)$ with $\mathcal{A} = \{A_1, \ldots, A_q\}$, and let $[a_0, a_1, \ldots, a_{N_1+N_2-1}] \in \mathcal{W}_{N_1+N_2}(\mathcal{C};Q)$. Then there exists $x \in Q$ with $\varphi(j\tau, x, v_{a_0,a_1,\ldots,a_{N_1+N_2-1}}) \in A_{a_j}$ for $j = 0, 1, \ldots, N_1 + N_2 - 1$. Let $y := \varphi(N_1\tau, x, v_{a_0,a_1,\ldots,a_{N_1+N_2-1}})$. Then $y \in Q$ and by the cocycle property (1.12)

$$\varphi\left(j\tau, y, v_{a_{N_1},a_{N_1+1},\ldots,a_{N_1+N_2-1}}\right) \in A_{a_{N_1+j}} \quad \text{for } j = 0, 1, \ldots, N_2 - 1.$$

This proves that $[a_{N_1}, a_{N_1+1}, \ldots, a_{N_1+N_2-1}]$ is an admissible word of length N_2. Hence, we can define α by

$$\alpha : [a_0, a_1, \ldots, a_{N_1+N_2-1}] \mapsto \left([a_0, a_1, \ldots, a_{N_1-1}], [a_{N_1}, \ldots, a_{N_1+N_2-1}]\right).$$

Injectivity of α is obvious. □

By the next proposition the strict invariance entropy of a pair (K, Q) with respect to an invariant covering is always an upper bound for $h^*_{\text{inv}}(K, Q)$.

5.1.5 Proposition:

Let Q be a compact controlled invariant set for control system (1.7) *and $K \subset Q$ compact. Then for any invariant covering $\mathcal{C} = (\mathcal{A}, v, \tau)$ of Q the following holds:*

$$\boxed{h^*_{\text{inv}}(K,Q) \leq h^*_{\text{inv}}(\mathcal{C};K,Q) \leq \frac{\ln \#\mathcal{A}}{\tau}.}$$

Proof:
Let $q = \#\mathcal{A}$. Since $\mathcal{W}_N(\mathcal{C};K,Q) \subset \{1,\ldots,q\}^N$, we have $\#\mathcal{W}_N(\mathcal{C};K,Q) \leq \#\{1,\ldots,q\}^N = q^N$ and thus

$$\frac{\ln \#\mathcal{W}_N(\mathcal{C};K,Q)}{N\tau} \leq \frac{\ln q^N}{N\tau} = \frac{\ln \#\mathcal{A}}{\tau} \quad \text{for all } N \in \mathbb{N}.$$

This implies $h^*_{\text{inv}}(\mathcal{C};K,Q) \leq \frac{\ln \#\mathcal{A}}{\tau}$. Now consider for every $N \in \mathbb{N}$ the set

$$\mathcal{S}_N := \left\{ v_{a_0,a_1,\ldots,a_{N-1}} \mid [a_0, a_1, \ldots, a_{N-1}] \in \mathcal{W}_N(\mathcal{C};K,Q) \right\}.$$

Let $\mathcal{A} = \{A_1, \dots, A_q\}$ and $v_a = v(A_a)$ for $a = 1, \dots, q$. We want to show that $\mathcal{S}_N$ is $(N\tau)$-spanning for (K, Q). To this end, pick $x_0 \in K$ arbitrarily. Then there exists $a_0 \in \{1, \dots, q\}$ with $x_0 \in A_{a_0}$. This implies

$$\varphi\left([0,\tau], x_0, v_{a_0}\right) \subset \varphi([0,\tau], A_{a_0}, v_{a_0}) \subset Q.$$

Let $x_1 := \varphi(\tau, x_0, v_{a_0})$. Then there is $a_1 \in \{1, \dots, q\}$ with $x_1 \in A_{a_1}$ and we obtain $\varphi([0,\tau], x_1, v_{a_1}) \subset Q$. Again, for $x_2 := \varphi(\tau, x_1, v_{a_1})$ we have $x_2 \in A_{a_2}$ for some a_2. Repeating this process, after N steps, we have found an admissible word $[a_0, a_1, \dots, a_{N-1}]$ for (K, Q), since the cocycle property implies

$$\varphi\left(j\tau, x_0, v_{a_0,a_1,\dots,a_{N-1}}\right) \in A_{a_j} \quad \text{for } j = 0, 1, \dots, N-1.$$

Hence, $\mathcal{S}_N$ is $(N\tau)$-spanning for (K, Q) and we obtain

$$r^*_{\text{inv}}(N\tau, K, Q) \leq \#\mathcal{W}_N(\mathcal{C}; K, Q) \quad \text{for all } N \in \mathbb{N},$$

which by Proposition 2.2.10 implies

$$\begin{aligned} h^*_{\text{inv}}(K,Q) &\overset{(2.14)}{=} \limsup_{N\to\infty} \frac{1}{N\tau} \ln r^*_{\text{inv}}(N\tau, K, Q) \\ &\leq \limsup_{N\to\infty} \frac{\ln \#\mathcal{W}_N(\mathcal{C}; K, Q)}{N\tau} \overset{(5.1)}{=} h^*_{\text{inv}}(\mathcal{C}; K, Q). \end{aligned}$$

This finishes the proof. □

The following lemma shows that, in order to approximate $h^*_{\text{inv}}(Q)$ by the quantities $h^*_{\text{inv}}(\mathcal{C}; Q)$, it is sufficient to consider invariant coverings $(\mathcal{A}, v, \tau)$, where $\mathcal{A}$ is a partition of the set Q.

5.1.6 Lemma:

For every invariant covering $\mathcal{C} = (\mathcal{A}, v, \tau)$ of a compact controlled invariant set Q there exists another invariant covering $\tilde{\mathcal{C}} = (\tilde{\mathcal{A}}, \tilde{v}, \tau)$ such that $\tilde{\mathcal{A}}$ is a partition of Q with $\#\tilde{\mathcal{A}} = \#\mathcal{A}$ and

$$h^*_{\text{inv}}(\tilde{\mathcal{C}}; Q) \leq h^*_{\text{inv}}(\mathcal{C}; Q).$$

Proof:
Let $\mathcal{A} = \{A_1, \dots, A_q\}$ and define sets $\tilde{A}_1, \dots, \tilde{A}_q$ by

$$\tilde{A}_1 := A_1, \quad \tilde{A}_j := A_j \backslash \bigcup_{i=1}^{j-1} A_i \quad \text{for} \quad j = 2, \dots, q.$$

Then $\tilde{\mathcal{A}} := \{\tilde{A}_1, \dots, \tilde{A}_q\}$ is a partition of Q, which is proved as follows. For $j_1 < j_2$ we have

$$\begin{aligned} \tilde{A}_{j_1} \cap \tilde{A}_{j_2} &= \left(A_{j_1} \backslash \bigcup_{i=1}^{j_1-1} A_i\right) \cap \left(A_{j_2} \backslash \bigcup_{i=1}^{j_2-1} A_i\right) \\ &= \left(A_{j_1} \cap \bigcap_{i=1}^{j_1-1} A_i^c\right) \cap \left(A_{j_2} \cap \bigcap_{i=1}^{j_2-1} A_i^c\right) \\ &= A_{j_1} \cap A_{j_2} \cap A_1^c \cap \dots \cap A_{j_1}^c \cap \dots \cap A_{j_2-1}^c = \emptyset. \end{aligned}$$

Hence, the elements of $\tilde{\mathcal{A}}$ are disjoint. If $x \in Q$, then $x \in A_{j(x)}$ with

$$j(x) := \min \{ j \in \{1, \ldots, q\} \mid x \in A_j \} .$$

This yields

$$x \in A_{j(x)} \setminus \bigcup_{i=1}^{j(x)-1} A_i = \tilde{A}_{j(x)}.$$

Hence, $\tilde{\mathcal{A}}$ is a partition of Q. By setting $\tilde{v}(\tilde{A}_j) := v(A_j)$, $j = 1, \ldots, q$, we obtain an invariant covering $\tilde{\mathcal{C}} = (\tilde{\mathcal{A}}, \tilde{v}, \tau)$, since $\tilde{A}_j \subset A_j$. Now let $[a_0, a_1, \ldots, a_{N-1}]$ be an admissible word of length N for $\tilde{\mathcal{C}}$. Then there exists $x \in Q$ with

$$\varphi \left(j\tau, x, \tilde{v}_{a_0,a_1,\ldots,a_{N-1}} \right) = \varphi \left(j\tau, x, v_{a_0,a_1,\ldots,a_{N-1}} \right) \in \tilde{A}_{a_j} \subset A_{a_j}$$

for $j = 0, 1, \ldots, N-1$. This implies that $[a_0, a_1, \ldots, a_{N-1}]$ is also admissible for $\mathcal{C}$ and hence $\mathcal{W}_N(\tilde{\mathcal{C}}; Q) \subset \mathcal{W}_N(\mathcal{C}; Q)$ for all $N \in \mathbb{N}$, which yields the assertion. □

As the next lemma shows, it is also sufficient to assume that $v : \mathcal{A} \to \mathcal{U}$ is injective (considered as a function from $\mathcal{A}$ to $\{u|_{[0,\tau]} : \ u \in \mathcal{U}\}$), i.e., that with every set A of the covering $\mathcal{A}$ a different control function is associated.

5.1.7 Lemma:
Let $\mathcal{C} = (\mathcal{A}, v, \tau)$ be an invariant covering of a compact controlled invariant set Q such that $\mathcal{A}$ is a partition of Q. Assume that $\mathcal{B}$ is a refinement of $\mathcal{A}$, i.e., a partition of Q such that for every $B \in \mathcal{B}$ there is an $A \in \mathcal{A}$ with $B \subset A$. Define $w : \mathcal{B} \to \mathcal{U}$ by $w(B) := v(A)$, where A is defined by the relation $B \subset A$. Then $\mathcal{D} := (\mathcal{B}, w, \tau)$ is an invariant covering of Q with

$$h^*_{\text{inv}}(\mathcal{D}; Q) \geq h^*_{\text{inv}}(\mathcal{C}; Q).$$

Proof:
It is clear that $\mathcal{D}$ is an invariant covering of Q. Let $[b_0, b_1, \ldots, b_{N-1}]$ be an admissible word for $\mathcal{D}$ and for each $j \in \{0, 1, \ldots, N-1\}$ let $a_j \in \{1, \ldots, \#\mathcal{A}\}$ be defined by the relation $B_{b_j} \subset A_{a_j}$. Then there exists $x \in Q$ with

$$\varphi \left(j\tau, x, v_{a_0,a_1,\ldots,a_{N-1}} \right) = \varphi \left(j\tau, x, w_{b_0,b_1,\ldots,b_{N-1}} \right) \in B_{b_j} \subset A_{a_j}.$$

Hence, $[a_0, a_1, \ldots, a_{N-1}]$ is an admissible word for $\mathcal{C}$ and thus for every $N \in \mathbb{N}$ we can define a mapping

$$\alpha_N : \mathcal{W}_N(\mathcal{D}; Q) \to \mathcal{W}_N(\mathcal{C}; Q), \quad [b_0, b_1, \ldots, b_{N-1}] \mapsto [a_0, a_1, \ldots, a_{N-1}].$$

This mapping is surjective, since for every admissible word $[a_0, a_1, \ldots, a_{N-1}]$ for $\mathcal{C}$ there is an $x \in Q$ with $\varphi(j\tau, x, v_{a_0,a_1,\ldots,a_{N-1}}) \in A_{a_j}$ for $j = 0, 1, \ldots, N-1$ and for every j there is b_j such that $\varphi(j\tau, x, v_{a_0,a_1,\ldots,a_{N-1}}) \in B_{b_j}$ and $B_{b_j} \subset A_{a_j}$. Hence, $[b_0, b_1, \ldots, b_{N-1}]$ is mapped to $[a_0, a_1, \ldots, a_{N-1}]$ by α_N. This implies

$$\#\mathcal{W}_N(\mathcal{C}; Q) \leq \#\mathcal{W}_N(\mathcal{D}; Q) \quad \text{for all } N \in \mathbb{N}.$$

Hence, $h^*_{\text{inv}}(\mathcal{C}; Q) \leq h^*_{\text{inv}}(\mathcal{D}; Q)$. □

5.1.8 Theorem (Characterization via Invariant Coverings):
Let Q be a compact controlled invariant set for control system (1.7). Then

$$\boxed{h^*_{\mathrm{inv}}(Q) = \inf_{\mathcal{C}} h^*_{\mathrm{inv}}(\mathcal{C};Q),} \tag{5.2}$$

where the infimum[1] is taken over all invariant coverings $\mathcal{C} = (\mathcal{A}, v, \tau)$ of Q such that $\mathcal{A}$ is a Borel measurable partition of Q and v is injective[2]. Moreover, it suffices to consider only times τ which are integer multiples of some $\tau_0 > 0$.

Proof:
For $h^*_{\mathrm{inv}}(Q) = \infty$ the assertion follows from Proposition 5.1.2. Hence, we may assume $h^*_{\mathrm{inv}}(Q) < \infty$. By Proposition 5.1.5 it suffices to show that there exists a sequence $(\mathcal{C}_k)_{k\in\mathbb{N}}$, $\mathcal{C}_k = (\mathcal{A}_k, v_k, \tau_k)$, of invariant coverings such that $\mathcal{A}_k$ is a measurable partition of Q, v_k is injective, $\tau_k = k\tau_0$ for some $\tau_0 > 0$ and $h^*_{\mathrm{inv}}(\mathcal{C}_k;Q) \to h^*_{\mathrm{inv}}(Q)$ for $k \to \infty$. To this end, fix $\tau_0 > 0$ and let $\tau_k := k\tau_0$. For each $k \in \mathbb{N}$ let $\mathcal{S}_k = \{u_1^k, \ldots, u_{n_k}^k\}$ be a minimal $k\tau_0$-spanning set for Q and define the covering $\tilde{\mathcal{A}}_k = \{\tilde{A}_1, \ldots, \tilde{A}_{n_k}\}$ by

$$\tilde{A}_j := \left\{x \in Q \ : \ \varphi\left([0,k\tau_0], x, u_j^k\right) \subset Q\right\}, \quad j = 1, \ldots, n_k.$$

$\tilde{A}_j$ is a G_δ-set for every $j \in \{1, \ldots, n_k\}$, which follows from the identity

$$\tilde{A}_j = \bigcap_{n\in\mathbb{N}} \left\{x \in Q \ : \ \varphi\left([0,k\tau_0], x, u_j^k\right) \subset N_{1/n}(Q)\right\}.$$

Now we construct a partition $\mathcal{A}_k$ from $\tilde{\mathcal{A}}_k$ by

$$A_1 := \tilde{A}_1, \quad A_j := \tilde{A}_j \backslash \bigcup_{i=1}^{j-1} \tilde{A}_i, \quad j = 2, \ldots, n_k.$$

By the proof of Lemma 5.1.6 the sets A_j form a measurable partition of Q. Let $v_k : \mathcal{A} \to \mathcal{U}$ be given by $v_k(A_j) := u_j^k$, $j = 1, \ldots, n_k$. Then $(\mathcal{A}_k, v_k, \tau_k)$ is obviously an invariant covering of Q and v_k is injective. By Proposition 2.2.10 and Proposition 2.1.10(iii) we have

$$h^*_{\mathrm{inv}}(Q) = \lim_{k\to\infty} \frac{1}{k\tau_0} \ln n_k = \inf_{k\in\mathbb{N}} \frac{1}{k\tau_0} \ln n_k.$$

Hence, for given $\varepsilon > 0$ we can choose $k_0 \in \mathbb{N}$ big enough such that $\frac{1}{k\tau_0} \ln n_k - h^*_{\mathrm{inv}}(Q) < \varepsilon$ for all $k \geq k_0$. Together with Proposition 5.1.5 we obtain

$$h^*_{\mathrm{inv}}(Q) \leq h^*_{\mathrm{inv}}(\mathcal{C}_k;Q) \leq \frac{\ln n_k}{k\tau_0} < h^*_{\mathrm{inv}}(Q) + \varepsilon \ \text{ for all } k \geq k_0.$$

This implies the assertion. □

---

[1] $\inf \emptyset$ is defined as ∞ (in the case there is no invariant covering).
[2] Here v is considered as a function from $\mathcal{A}$ to $\{u|_{[0,\tau]} \ : \ u \in \mathcal{U}\}$.

5.1.9 Remark:
Note that the proof of Theorem 5.1.8 does not work for the case $K \neq Q$, since then a T-spanning set for (K, Q) does not yield an invariant covering of Q.

5.1.10 Remark:
The characterization of the strict invariance entropy, given in Theorem 5.1.8, essentially coincides with the definition of *(strong) topological feedback entropy*, introduced in Nair & Evans & Mareels & Moran [42]. But nevertheless there are crucial differences between the two notions. Topological feedback entropy is defined for discrete-time control systems of the form

$$x_{k+1} = F(x_k, u_k), \quad k \geq 0,$$

where the state space X is a topological space and the controls u_k are taken from an arbitrary set U. A compact set $Q \subset X$ with nonvoid interior is considered such that there is another compact set $Q' \subset \operatorname{int} Q$, and the following holds: For every $x_0 \in Q$ there is a control $u_0 \in U$ with $x_1 = F(x_0, u_0) \in \operatorname{int} Q'$. This invariance condition—called *strong invariance* in [42]—differs from the controlled invariance that we impose on the set Q. For example, if Q is the closure of a variant control set with nonvoid interior, then there are always points on the boundary of Q which cannot be steered to the interior. The strong invariance condition in [42], which is tailored for stabilization problems, also makes it possible to consider only open covers of Q.

5.1.11 Example:
Consider the one-dimensional linear system (3.15) from Example 3.1.7 with $a > 0$ and control range $U = [u_{\min}, u_{\max}]$. Let $Q = [q_1, q_2] \subset \frac{1}{a}[-u_{\max}, -u_{\min}]$ with $q_1 < q_2$. Then Q is a controlled invariant compact interval, since every point in Q becomes an equilibrium for some constant control function. We define an invariant covering $(\mathcal{A}, v, \tau)$ of Q as follows: Let $p := \frac{q_1+q_2}{2}$ and $A_1 := [q_1, p]$, $A_2 := [p, q_2]$. Then $\mathcal{A} := \{A_1, A_2\}$ is a covering of Q. Let $v_1(t) :\equiv v(A_1)(t) :\equiv -aq_1$ and $v_2(t) :\equiv v(A_2)(t) :\equiv -aq_2$. Finally, let $\tau := \frac{\ln(2)}{a}$. Then $(\mathcal{A}, v, \tau)$ is an invariant covering, since for all $t \in [0, \tau]$ we have

$$\begin{aligned}
\varphi(t, A_1, v_1) &= e^{at} A_1 - aq_1 \int_0^t e^{a(t-s)} ds = \left[e^{at} q_1, e^{at} p\right] + q_1(1 - e^{at}) \\
&= \left[q_1, q_1 + e^{at} \tfrac{q_2 - q_1}{2}\right] \subset \left[q_1, q_1 + e^{a\tau} \tfrac{q_2 - q_1}{2}\right] = [q_1, q_2] = Q
\end{aligned}$$

and

$$\begin{aligned}
\varphi(t, A_2, v_2) &= e^{at} A_2 - aq_2 \int_0^t e^{a(t-s)} ds = \left[e^{at} p, e^{at} q_2\right] + q_2(1 - e^{at}) \\
&= \left[q_2 - e^{at} \tfrac{q_2 - q_1}{2}, q_2\right] \subset \left[q_2 - e^{a\tau} \tfrac{q_2 - q_1}{2}, q_2\right] = [q_1, q_2] = Q.
\end{aligned}$$

Now Proposition 5.1.5 implies that $h^*_{\mathrm{inv}}(Q)$ is bounded from above by $\frac{\ln \#\mathcal{A}}{\tau} = a$, and together with $h^*_{\mathrm{inv}}(Q) \geq h_{\mathrm{inv}}(Q) = a$ (see Example 3.2.10) we obtain $h^*_{\mathrm{inv}}(Q) = a$.

Now we consider system (3.15) with $a \leq 0$: Let $Q = [q_1, q_2]$ be any compact controlled invariant interval. Again we define $p := \frac{q_1+q_2}{2}$. Let $u(t) :\equiv -ap$. Then for all $t \geq 0$ we have

$$\varphi(t, q_1, u) = e^{at}q_1 + \tfrac{q_1+q_2}{2}(1 - e^{at}) = \tfrac{q_1+q_2}{2} - e^{at}\tfrac{q_2-q_1}{2} \subset [q_1, p] \subset Q,$$

since $e^{at} \in [0,1]$ for $t \geq 0$. Analogously we get $\varphi(t, q_2, u) \in Q$ for all $t \geq 0$. Hence, $r^*_{\text{inv}}(T, Q) = 1$ for all $T > 0$, which implies $h^*_{\text{inv}}(Q) = 0$.

Thus, we have the result

$$\boxed{h^*_{\text{inv}}(Q) = \max\{0, a\},}$$

which also shows that the strict invariance entropy coincides with the invariance entropy in this case (see Example 3.2.10). ◊

5.1.12 Open Questions:

- Are there smaller families of invariant coverings which are sufficient to approximate $h^*_{\text{inv}}(Q)$?
- Can Theorem 5.1.8 be generalized for the case $K \neq Q$?

5.2 Relation to Data Rates

In the following, we prove that the strict invariance entropy $h^*_{\text{inv}}(Q)$ coincides with the infimum data rate necessary to render the set Q invariant by a causal coding and control law. To this end, we use the characterization via invariant coverings and adapt the corresponding proof for the topological feedback entropy (see Nair & Evans & Mareels & Moran [42, Theorem 1, p. 1588]).

Consider control system (1.7) and suppose that a sensor, which is connected to a controller via a digital noiseless channel, measures its states at sampling times $k\tau$, $k \in \mathbb{N}_0$, for some fixed time step $\tau > 0$. The state at time $k\tau$ is coded using a finite coding alphabet S_k of (time-varying) size μ_k. We require that the sequence $(\mu_k)_{k\in\mathbb{N}_0}$ satisfies

$$\liminf_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} \log_2 \mu_j < \infty. \tag{5.3}$$

The coder transmits a symbol $s_k \in S_k$, which may depend on the present state and on all past states. The corresponding coder mapping is denoted by

$$\gamma_k : M^{k+1} \to S_k.$$

At time $k\tau$ the controller has $k+1$ symbols $s_0, s_1, \ldots, s_k$ available and generates a finite-time control function $u_k : [0, \tau] \to U$. We denote the corresponding controller mapping by

$$\delta_k : S_0 \times S_1 \times \cdots \times S_k \to \mathcal{U}_\tau := \left\{u|_{[0,\tau]} : \ u \in \mathcal{U}\right\}.$$

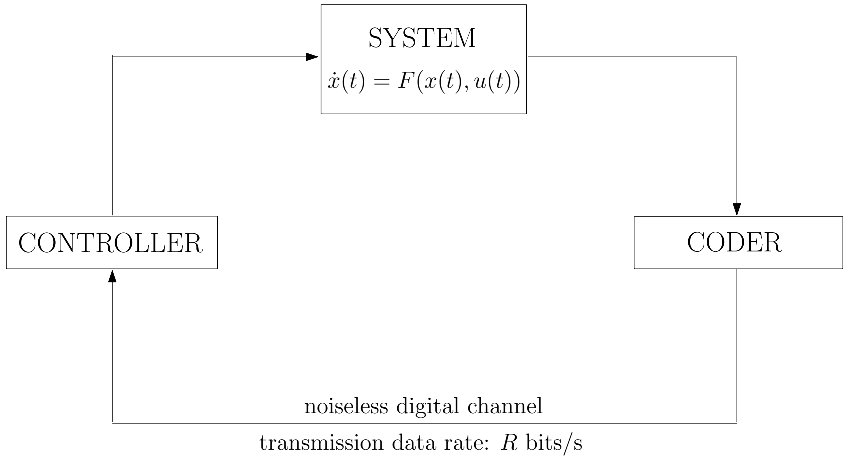

Figure 5.1: A Coder-Controller

5.2.1 Definition (Coder-Controller):
The 4-tuple $\mathcal{H} := (S, \gamma, \delta, \tau)$, where $S = (S_k)_{k\in\mathbb{N}_0}$, $\gamma = (\gamma_k)_{k\in\mathbb{N}_0}$, and $\delta = (\delta_k)_{k\in\mathbb{N}_0}$, is called a **coder-controller**. *We define the* **transmission data rate** *of $\mathcal{H}$ by*

$$R(\mathcal{H}) := \liminf_{k\to\infty} \frac{1}{k\tau} \sum_{j=0}^{k-1} \log_2 \mu_j, \tag{5.4}$$

which by (5.3) is a finite number.[3] *We say that* **$\mathcal{H}$ renders Q invariant** *if for all $x_0 \in Q$ the sequence $(x_k)_{k\in\mathbb{N}_0}$, defined recursively by*

$$x_k := \varphi\left(\tau, x_{k-1}, u_{k-1}\right)$$

with

$$u_{k-1} = \delta_{k-1}(\gamma_0(x_0), \gamma_1(x_0, x_1), \ldots, \gamma_{k-1}(x_0, x_1, \ldots, x_{k-1})),$$

satisfies

$$\varphi([0,\tau], x_k, u_k) \subset Q \quad \text{for all } k \in \mathbb{N}_0.$$

That is, starting in Q at time $k = 0$ one stays in Q forever.

5.2.2 Theorem:
Let Q be a compact controlled invariant set for control system (1.7). Then

$$\boxed{h^*_{\text{inv}}(Q) = \inf_{\mathcal{H}} \frac{R(\mathcal{H})}{\log_2 e},} \tag{5.5}$$

where the infimum is taken over all coder-controllers $\mathcal{H}$, which render Q invariant.[4]

[3] The definition of the transmission data rate is taken from Nair & Evans & Mareels & Moran [42]. Probably it would not make much difference if we consider the limes superior instead of the limes inferior in this definition.

[4] As in Theorem 5.1.8, $\inf \emptyset$ is defined as ∞ (in the case there is no coder-controller rendering Q invariant).

Proof:

The proof proceeds in three steps.

Step 1: Assume that $h^*_{\text{inv}}(Q) = \infty$ and let $\mathcal{H} = (S, \gamma, \delta, \tau)$ be a coder-controller rendering Q invariant. Consider the sets

$$A_s := \{x \in Q \mid \varphi([0,\tau], x, \delta_0(s)) \subset Q\}, \quad s \in S_0.$$

The family $\mathcal{A} := \{A_s\}_{s \in S_0}$ is a finite covering of Q, since $x \in A_{\delta_0(\gamma_0(x))}$ holds by Definition 5.2.1. With $v(A_s) := \delta_0(s)$, $s \in S_0$, one obtains an invariant covering $(\mathcal{A}, v, \tau)$ of Q. By Proposition 5.1.2 this contradicts $h^*_{\text{inv}}(Q) = \infty$. Hence, the assertion holds if $h^*_{\text{inv}}(Q) = \infty$.

Step 2: For an arbitrary coder-controller $\mathcal{H} = (S, \gamma, \delta, \tau)$, rendering Q invariant, we show that $\frac{R(\mathcal{H})}{\log_2 e} \geq h^*_{\text{inv}}(Q)$: It immediately follows from the definition of $R(\mathcal{H})$ that for given $\varepsilon > 0$ there exists $r \in \mathbb{N}$ such that

$$\frac{1}{r\tau} \sum_{k=0}^{r-1} \log_2 \mu_k < R(\mathcal{H}) + \varepsilon \log_2 e. \tag{5.6}$$

For every sequence $(s_0, s_1, \ldots, s_{r-1}) \in S_0 \times S_1 \times \cdots \times S_{r-1}$ let

$$A_{s_0, s_1, \ldots, s_{r-1}} := \{x_0 \in Q \mid \gamma_j(x_0, \ldots, x_j) = s_j \text{ for } j = 0, 1, \ldots, r-1\},$$

where $x_0, x_1, \ldots, x_{r-1}$ are defined as in Definition 5.2.1. Then the family $\mathcal{A}$ of all the sets $A_{s_0, s_1, \ldots, s_{r-1}}$ obviously is a finite covering of Q, which can be extended to an invariant covering $\mathcal{C} = (\mathcal{A}, v, r\tau)$, where v assigns to the set $A_{s_0, s_1, \ldots, s_{r-1}}$ the control function given by concatenation of $u_0, u_1, \ldots, u_{r-1}$, which are defined as in Definition 5.2.1. By Proposition 5.1.5 we obtain

$$\begin{aligned} h^*_{\text{inv}}(Q) \;\leq\; h^*_{\text{inv}}(\mathcal{C}; Q) \leq \frac{\ln \#\mathcal{A}}{r\tau} &= \frac{\ln \prod_{k=0}^{r-1} \mu_k}{r\tau} = \frac{\sum_{k=0}^{r-1} \ln \mu_k}{r\tau} \\ &= \frac{1}{\log_2 e} \left[\frac{1}{r\tau} \sum_{k=0}^{r-1} \log_2 \mu_k \right] \overset{(5.6)}{<} \frac{R(\mathcal{H})}{\log_2 e} + \varepsilon. \end{aligned}$$

Since this holds for every $\varepsilon > 0$, the assertion follows.

Step 3: We show that there exist coder-controllers rendering Q invariant, whose transmission data rates come arbitrarily close to $(\log_2 e) h^*_{\text{inv}}(Q)$: By the proof of Theorem 5.1.8 there exists a sequence $(\mathcal{C}_n)_{n \in \mathbb{N}}$, $\mathcal{C}_n = (\mathcal{A}_n, v_n, \tau_n)$, $\mathcal{A}_n = \{A^n_1, \ldots, A^n_{q_n}\}$, of invariant coverings of Q such that $\mathcal{A}_n$ is a partition of Q and

$$\frac{\ln q_n}{\tau_n} \to h^*_{\text{inv}}(Q) \;\text{ for } n \to \infty. \tag{5.7}$$

Define the coder-controller $\mathcal{H}^n = (S^n, \gamma^n, \delta^n, \tau^n)$ by

$$\begin{aligned} \tau^n &:= \tau_n, \\ S^n_k &:= \{1, \ldots, q_n\}, \\ \gamma^n_k(x_0, x_1, \ldots, x_k) &:= s_k \text{ with } x_k \in A^n_{s_k}, \\ \delta^n_k(s_0, s_1, \ldots, s_k) &:= v_n(A^n_{s_k})|_{[0,\tau_n]} \end{aligned}$$

for all $k \in \mathbb{N}_0$ and for each $n \in \mathbb{N}$.[5] From the definition of invariant coverings it immediately follows that $\mathcal{H}^n$ renders Q invariant. For the corresponding transmission data rates we obtain

$$R(\mathcal{H}^n) = \liminf_{k\to\infty} \frac{1}{k\tau_n} \underbrace{\sum_{j=0}^{k-1} \log_2 q_n}_{=k\log_2 q_n} = \frac{\log_2 q_n}{\tau_n} = (\log_2 e)\,\frac{\ln q_n}{\tau_n} \stackrel{(5.7)}{\longrightarrow} (\log_2 e) h^*_{\mathrm{inv}}(Q).$$

This completes the proof. □

5.2.3 Remarks:

- Note that the factor $\log_2 e$ only appears in Formula (5.5) since we use the natural logarithm instead of the logarithm with base 2 in the definition of invariance entropy.
- In Nair & Evans & Mareels & Moran [42] it is only required that the sequence $(\mu_k)_{k\in\mathbb{N}_0}$ of alphabet sizes satisfies $\frac{1}{k}\log_2 \mu_k \to 0$ for $k \to \infty$, i.e., that it grows subexponentially, which also allows infinite transmission data rates, e.g., if μ_k grows linearly.[6]
- In Nair & Evans & Mareels & Moran [42] the symbols generated by the coder are also allowed to depend on the past symbols. But since the past symbols can be generated from the past states, we only consider coders with inputs from the state space.

5.2.4 Open Question:

Is there a generalization of Theorem 5.2.2 for the case $K \neq Q$?

5.3 Notes on Numerical Computation

In this section, we sketch a numerical algorithm for computing rigorous upper bounds of the strict invariance entropy $h^*_{\mathrm{inv}}(Q)$, based on the characterization via invariant coverings. The algorithm splits into two main tasks:

(i) Construction of a "good" invariant covering $\mathcal{C}$ of Q.

(ii) Computation of $h^*_{\mathrm{inv}}(\mathcal{C}; Q)$.

By Theorem 5.1.8 $h^*_{\mathrm{inv}}(\mathcal{C}; Q)$ is an upper bound for $h^*_{\mathrm{inv}}(Q)$, and if the invariant covering is chosen appropriately, it comes arbitrarily close to $h^*_{\mathrm{inv}}(Q)$ (that is what we mean by "good"). In the following, we explain how these tasks can be realized.

[5] Hence, in particular the alphabet of $\mathcal{H}^n$ is time-invariant and the coder and controller mappings depend only on the present state or symbol, respectively.

[6] For $\mu_k \equiv k+1$ one has $\frac{1}{k}\sum_{j=0}^{k-1}\log_2(j+1) = \frac{1}{k}\log_2(k!)$. Since $(2k)! \geq k^k k!$, one obtains $\frac{1}{2k}\log_2((2k)!) \geq \frac{1}{2}\log_2(k) + \frac{1}{2k}\log_2(k!) \to \infty$.

(I) Construction of a Good Invariant Covering

By a good invariant covering we mean a covering $\mathcal{C}$, which produces a slow increase in the number of admissible words (such that the approximation is good). On the other hand, it should also result in a reasonable computation time for $h^*_{\text{inv}}(\mathcal{C};Q)$. Numerically we cannot work with arbitrary admissible control functions. So we restrict ourselves to piecewise constant ones. The idea for the construction of the invariant covering $\mathcal{C}$ is as follows: We first choose a coarse partition $\mathcal{A}_0 = \{A^0_1, \ldots, A^0_q\}$ of Q (the coarser the better), and a small time step $\tau_0 > 0$, such that $(\mathcal{A}_0, \tau_0)$ can be extended to an invariant covering of Q by assigning constant control functions to the sets in $\mathcal{A}_0$. (We have to impose the assumption on Q that this is possible.) From Proposition 5.1.5 we know that

$$h^*_{\text{inv}}(Q) \leq \frac{\ln \#\mathcal{A}_0}{\tau_0} \tag{5.8}$$

must hold. If we know an upper bound $L > 0$ for $h^*_{\text{inv}}(Q)$, then choosing

$$\tau_0 \leq \frac{\ln \#\mathcal{A}_0}{L}$$

guarantees that (5.8) holds. Analogously, any lower bound for $h^*_{\text{inv}}(Q)$ leads to a necessary condition for the choice of τ_0. When τ_0 is chosen, we check if $\varphi(\tau_0, A^0_i, u) \subset Q$ holds for all $i = 1, \ldots, q$ and all constant control functions u taking values in a finite uniformly distributed set $\widehat{U} \subset U$. Note that, if Q has the no-return property, then $\varphi(\tau_0, A^0_i, u) \subset Q$ guarantees that $\varphi([0,\tau_0], A^0_i, u) \subset Q$. Otherwise it is possible that $\varphi(t, A^0_i, u) \cap Q^c \neq \emptyset$ for some $t \in (0,\tau)$. In this case, we can estimate the maximal distance from a point $z \in \varphi(t, A^0_i, u)$ to the set Q, and choose a smaller value for τ_0, if we do not want to accept this error. If $\delta > 0$ is the maximal distance we accept, then we have to choose

$$\tau_0 \leq \frac{\delta}{\sup_{(z,v)\in N_\delta(Q)\times U} \|F(z,v)\|},$$

since $\sup_{(z,v)\in N_\delta(Q)\times U} \|F(z,v)\|$ is the maximal speed at which a trajectory in $N_\delta(Q)$ can move away from Q. Assume that for each of the sets A^0_i we find at least one control $u \in \widehat{U}$ such that $\varphi([0,\tau_0], A^0_i, u) \subset Q$ (approximately) holds. (Otherwise we have to start again with a finer partition and/or a smaller time step and/or a bigger set $\widehat{U}$ of allowed control values.) Then we can assign the maximal (nonempty) set $\widehat{U}_i \subset \widehat{U}$ to each $i \in \{1, \ldots, q\}$ such that

$$\varphi\left([0,\tau_0], A^0_i, u\right) \subset Q \quad \text{for all } u \in \widehat{U}_i.$$

Let

$$\widehat{U}_i = \{u_{i1}, \ldots, u_{in_i}\}, \quad n_i \in \mathbb{N}.$$

Now we choose a number $K \in \mathbb{N}$ and set $\tau := K\tau_0$, which will be the time step for our final invariant covering $(\mathcal{A}, v, \tau)$, whose entropy will be computed and used as an approximation for $h^*_{\text{inv}}(Q)$. For every set A^0_i, $i = 1, \ldots, q$, and every control u_{ij}, $j = 1, \ldots, n_i$, we check which of the intersections $\varphi(\tau_0, A^0_i, u_{ij}) \cap A^0_k$,

$k = 1, \dots, q$, is nonempty. This defines for every $i \in \{1, \dots, q\}$ a $q \times n_i$-transition matrix $M_i = (m^i_{jk})$, where

$$m^i_{jk} = \begin{cases} 1 & \text{if } \varphi(\tau_0, A^0_i, u_{ij}) \cap A^0_k \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Then for each of the sets $A^0_{i_0}$, $i_0 = 1, \dots, q$, we know possible control functions, defined on the time interval $[0, \tau]$, of the form

$$u = u_{i_0 j_1} \cdot u_{i_1 j_2} \cdot \dots \cdot u_{i_{K-1} j_K}, \quad \text{(concatenation)}$$

where $i_1, \dots, i_{K-1} \in \{1, \dots, q\}$, $j_l \in \{1, \dots, n_{i_{l-1}}\}$, and

$$m^{i_0}_{j_1 i_1} = 1, \quad m^{i_1}_{j_2 i_2} = 1, \dots, m^{i_{K-2}}_{j_{K-1} i_{K-1}} = 1.$$

Let $\overline{\mathcal{U}}_i$ denote the set of these control functions for A^0_i. Then for each $x \in A^0_i$ there is at least one $u \in \overline{\mathcal{U}}_i$ such that $\varphi(j\tau_0, x, u) \in Q$ for $j = 1, \dots, K$. Now we subdivide the sets A^0_i into smaller sets $A_{ij} \subset A^0_i$, $j = 1, \dots, m_i$, such that for each A_{ij} we find some $u_{ij} \in \overline{\mathcal{U}}_i$ with $\varphi(j\tau_0, A_{ij}, u_{ij}) \subset Q$ for $j = 1, \dots, K$. Then we define $\mathcal{A}$ as the family consisting of all the sets A_{ij}, $(i, j) \in \{1, \dots, q\} \times \{1, \dots, m_i\}$, and we assign to each A_{ij} a control function $v(A_{ij})$ of the u_{ij}'s, such that the number of sets hit by $\varphi(\tau, A_{ij}, v(A_{ij}))$ becomes as small as possible. This defines our invariant covering $\mathcal{C} = (\mathcal{A}, v, \tau)$.

The proof of Theorem 5.1.8 suggests that $h^*_{\text{inv}}(\mathcal{C}; Q)$ comes closer to $h^*_{\text{inv}}(Q)$ if $\tau = K\tau_0$ becomes bigger. It is also reasonable to expect that τ_0 should be small and $\#\widehat{U}$ big in order to approximate $h^*_{\text{inv}}(Q)$ more accurately, since then more admissible control functions can be realized. How the number of elements in the initial partition $\mathcal{A}_0$ affects the entropy $h^*_{\text{inv}}(\mathcal{C}; Q)$ is not clear, but it should be kept small for the sake of a feasible expense of computational time and memory, especially since the refined partition $\mathcal{A}$ has to be refined again for the computation of $h^*_{\text{inv}}(\mathcal{C}; Q)$ and a lot of computations have to be performed on a transition matrix corresponding to that second refinement.

(II) Computation of the Entropy of an Invariant Covering

If $\mathcal{C} = (\mathcal{A}, v, \tau)$ is an invariant covering of the set Q such that $\mathcal{A} = \{A_1, \dots, A_q\}$ is a partition of Q, we can define a piecewise continuous map $f_{\mathcal{C}} : Q \to Q$ by

$$f_{\mathcal{C}}(x) := \varphi\left(\tau, x, v_{a(x)}\right), \text{ where } a(x) \text{ is defined by } x \in A_{a(x)}.$$

Now the sets $\mathcal{W}_N(\mathcal{C}; Q)$ of admissible words can be described in terms of the iterates of $f_{\mathcal{C}}$ as follows: A word $[a_0, a_1, \dots, a_{N-1}]$ is admissible for Q if and only if there exists $x \in Q$ with

$$x \in A_{a_0}, f_{\mathcal{C}}(x) \in A_{a_1}, \dots, f^{N-1}_{\mathcal{C}}(x) \in A_{a_{N-1}}.$$

By this observation, it turns out that the quantity $h^*_{\text{inv}}(\mathcal{C}; Q)$ coincides with the topological entropy of $f_{\mathcal{C}}$ with respect to the partition $\mathcal{A}$, as defined in Froyland & Junge & Ochs [21] (up to the factor τ). The exact definition is as follows.

5.3.1 Definition:
Let X be a set and $f : X \to X$ a map. Let $\mathcal{A} = \{A_1, \ldots, A_q\}$ be a finite partition of X. For every $N \in \mathbb{N}$ define

$$\mathcal{W}_N(f, \mathcal{A}) := \left\{[a_0, a_1, \ldots, a_{N-1}] \; : \; \exists x \in X : \; f^i(x) \in A_{a_i}, \; 0 \leq i < N\right\}.$$

Then the **entropy of f with respect to the partition $\mathcal{A}$** *is given by*

$$h^*(f, \mathcal{A}) := \lim_{N \to \infty} \frac{\ln \#\mathcal{W}_N(f, \mathcal{A})}{N}. \tag{5.9}$$

Hence, we can just use the algorithm presented in [21], once we have constructed an invariant covering. The basic idea of that algorithm is to define a topological Markov chain corresponding to a refinement $\mathcal{B}$ of the initial partition $\mathcal{A}$, which is regarded as a directed graph. The edges of this graph are endowed with labels corresponding to the initial partition $\mathcal{A}$. The words formed by traversing this labeled graph form a sofic shift, whose entropy is an upper bound for $h^*(f, \mathcal{A})$. The algorithm computes a reduced right-resolving representation of the sofic shift, which makes it possible to obtain the entropy as the maximal eigenvalue of a transition matrix. The upper bounds computed in this way converge to $h^*(f, \mathcal{A})$ as the diameter of the refinement $\mathcal{B}$ approaches zero.

On the complexity of this algorithm only little seems to be known. We refer to Froyland & Junge & Ochs [21] for further details.

Appendix

In the appendix, we present a compilation of central notions and results used in this thesis. The first section deals with differentiable manifolds and structures defined on them, including vector fields, Riemannian metrics and volume forms. Note that not every result here is presented with a reference. In the second section, we introduce the notions of fractal dimension and topological entropy, and in the third section we compile a couple of technical lemmas.

A.1 Manifolds

In this section, we recall some of the basic notions from differential topology and Riemannian geometry. We only consider C^∞-manifolds, which results in no loss of generality, since every C^k-differentiable structure on a manifold M contains a C^∞-differentiable structure (see Hirsch [29, Theorem 2.9, p. 51]). Note that for a C^∞-manifold M the tangent space T_pM at a point p can be defined as the vector space of derivations acting on functions germs defined in a neighborhood of p, which is not possible for C^k-manifolds with $k < \infty$ (see, e.g., Jurdjevic [32, p. 11]).

Smooth Manifolds

Let M be a second countable Hausdorff space. A family $\mathcal{A} = \{(\phi_\alpha, U_\alpha)\}_{\alpha \in A}$ is called a $\boldsymbol{C^\infty}$**-atlas** on M if the following axioms are satisfied:

(i) $\{U_\alpha\}_{\alpha \in A}$ is an open cover of M.

(ii) For each $\alpha \in A$, $\phi_\alpha : U_\alpha \to V_\alpha$ is a homeomorphism onto an open subset V_α of $\mathbb{R}^d$ for some $d \in \mathbb{N}$.

(iii) For all $\alpha, \beta \in A$ the transition function

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is a C^∞-diffeomorphism.

The space M together with the atlas $\mathcal{A}$ is called a $\boldsymbol{C^\infty}$**-manifold** or a **smooth manifold**. The elements (ϕ_α, U_α) of $\mathcal{A}$ are called **charts**, and the inverse functions $\phi_\alpha^{-1} : V_\alpha \to U_\alpha$ **local parametrizations** of M. Every C^∞-atlas $\mathcal{A}$

is contained in a unique maximal C^∞-atlas $\mathcal{A}_{\max}$.[7] If M is connected, then the natural number d (the dimension of the Euclidean space where ϕ_α takes its values) is independent of the chart. In this case, d is called the **dimension** of M and we write $d = \dim(M)$. In the following, we will assume that every manifold is connected and thus has a well-defined dimension. Every connected manifold is also path-connected. Moreover, every manifold is locally compact, locally path-connected and metrizable. When speaking of $\mathbb{R}^d$ as a smooth manifold we mean $\mathbb{R}^d$ together with the atlas consisting of the single chart $(\mathrm{id}_{\mathbb{R}^d}, \mathbb{R}^d)$. Every open subset N of a d-dimensional smooth manifold M with atlas $\mathcal{A}$ is itself a d-dimensional manifold, whereas an atlas for N is given by $\{(\phi|_{U\cap N}, U \cap N) \mid (\phi, U) \in \mathcal{A}\}$.

Let $f : M \to N$ be a continuous map between smooth manifolds M and N. Then f is called a $\boldsymbol{C^k}$**-map** ($k \in \mathbb{N} \cup \{\infty\}$) if for every $p \in M$ there exist charts (ϕ, U) of M and (ψ, V) of N with $p \in U$ and $f(p) \in V$ such that $f(U) \subset V$ and

$$\psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V)$$

is of class C^k. This definition is independent of the chosen charts. If f is invertible and both f and f^{-1} are C^k-maps, then f is called a $\boldsymbol{C^k}$**-diffeomorphism**. $C^k(M)$ denotes the set of all real-valued C^k-functions on a manifold M.

Let M be a d-dimensional smooth manifold and $p \in M$. Then $C^\infty(M, p)$ denotes the set of all real-valued C^∞-functions, defined on an open neighborhood of p, modulo the equivalence relation which identifies two functions if they coincide on some neighborhood of p. The set $C^\infty(M, p)$ has the canonical structure of both a real vector space and a commutative ring. A **tangent vector** of M at p is a linear map $\alpha : C^\infty(M, p) \to \mathbb{R}$ with the additional property[8]

$$\alpha(f \cdot g) = \alpha(f) \cdot g(p) + f(p) \cdot \alpha(g) \quad \text{for all } f, g \in C^\infty(M, p).$$

The **tangent space** T_pM is the set of all tangent vectors at p. T_pM with its canonical vector space structure is isomorphic to $\mathbb{R}^d$. If (ϕ, U) is a chart of M with $p \in U$, then a basis of T_pM is given by the vectors $\partial_1\phi_p, \dots, \partial_d\phi_p$, which are defined as follows:

$$(\partial_i\phi_p)(f) = \frac{\partial(f \circ \phi^{-1})}{\partial x_i}(\phi(p)) \quad \text{for all } f \in C^\infty(M, p).$$

Let $c : (-\varepsilon, \varepsilon) \to M$ be a curve with $c(t) = p$, which is differentiable at t. Then c defines a tangent vector $\dot{c}(t) \in T_pM$, given by

$$\dot{c}(t)(f) := \frac{d}{dt}(f \circ c)(t) \quad \text{for all } f \in C^\infty(M, p).$$

It can be shown that the following identity holds:

$$T_pM = \left\{\dot{c}(0) \mid c : (-\varepsilon, \varepsilon) \to M \ C^1\text{-curve with } c(0) = p\right\}.$$

[7]Maximality means that no further charts can be added to $\mathcal{A}_{\max}$ without destroying the property that all transition functions are of class C^∞.

[8]A function with this property is called a *derivation*.

If $f : M \to N$ is a C^k-map between manifolds M and N, then the derivative $Df(p) = Df_p : T_pM \to T_{f(p)}N$ of f at $p \in M$ is defined by

$$Df_p(\alpha)(\varphi) = \alpha(\varphi \circ f) \text{ for all } \alpha \in T_pM,\ \varphi \in C^\infty(N,p).$$

Df_p is a linear map and the following properties are satisfied:

(i) If f is a C^1-diffeomorphism, then Df_p is an isomorphism for all $p \in M$.

(ii) $D(g \circ f)(p) = Dg_{f(p)} \circ Df_p$ for C^k-maps $f : M \to N$ and $g : N \to P$.

(iii) $Df_p(\dot{c}(0)) = \frac{d}{dt}(f \circ c)(0)$ for a C^1-curve $c : (-\varepsilon, \varepsilon) \to M$ with $c(0) = p$.

The **tangent bundle** TM is the disjoint union of all tangent spaces of M. It can be equipped with a C^∞-atlas in a canonical way such that it becomes a $(2d)$-dimensional smooth manifold. In order to describe the charts of TM, consider the map $\pi : TM \to M$, $\pi(p,\alpha) = p$. If (ϕ, U) is a chart of M, then the associated chart of TM is $(\Phi, \pi^{-1}(U))$ with

$$\Phi(p,\alpha) = (\phi(p), D\phi_p\alpha) \in \mathbb{R}^d \times T_{\phi(p)}\mathbb{R}^d \cong \mathbb{R}^{2d}.$$

Note that Φ is a C^∞-map and that $T_{\phi(p)}\mathbb{R}^n$ can be identified with $\mathbb{R}^d$ canonically using the basis induced by the chart $(\mathrm{id}_{\mathbb{R}^d}, \mathbb{R}^d)$.

For every $p \in M$ we denote by T_p^*M the dual space of T_pM, $T_p^*M = \mathrm{Hom}(T_pM, \mathbb{R})$. The disjoint union T^*M of all these dual spaces is called the **cotangent bundle** of M. If (ϕ, U) is a chart of M and $p \in U$, then a basis $d\phi_1(p), \ldots, d\phi_d(p)$ of T_p^*M is given by

$$d\phi_i(p)(\alpha) = (D\phi_i)_p(\alpha),$$

where $\phi_i : U \to \mathbb{R}$ is the i^{th} coordinate function of ϕ and $(D\phi_i)_p : T_pM \to T_{\phi_i(p)}\mathbb{R} \cong \mathbb{R}$. $\{d\phi_1(p), \ldots, d\phi_d(p)\}$ is the dual basis of $\{\partial_1\phi(p), \ldots, \partial_d\phi(p)\}$, i.e., $d\phi_i(p)(\partial_j\phi(p)) = \delta_{ij}$.

A C^k-map $f : M \to N$ between manifolds M and N is called a $\boldsymbol{C^k}$**-submersion** if $Df_p : T_pM \to T_{f(p)}N$ is surjective for all $p \in M$. It is called a $\boldsymbol{C^k}$**-immersion** if Df_p is injective for all $p \in M$.

Vector Fields

A C^k-vector field ($k \in \mathbb{N}_0 \cup \{\infty\}$) on a d-dimensional smooth manifold M is a C^k-map $X : M \to TM$ such that $\pi \circ X = \mathrm{id}_M$, i.e., $X_p := X(p) \in T_pM$ for all $p \in M$. The vector field X is of class C^k if and only if for every chart (ϕ, U) of M the coefficient functions $\xi^1, \ldots, \xi^d : U \to \mathbb{R}$, defined by the equation

$$X_p = \sum_{i=1}^{d} \xi^i(p)\partial_i\phi_p \text{ for all } p \in U,$$

are of class C^k. The set $\mathcal{X}^k(M)$ of all C^k-vector fields on M has the canonical structure of a real vector space.

Every $X \in \mathcal{X}^\infty(M)$ induces a map on $C^\infty(M)$, also denoted by X, given by

$$X(f)(p) := X_p(f) \quad \text{for all } f \in C^\infty(M) \text{ and } p \in M,$$

where on the right-hand side of the equation above f is considered to be an element of $C^\infty(M, p)$. X is completely determined by its induced map on $C^\infty(M)$. For two vector fields $X, Y \in \mathcal{X}^\infty(M)$ a third vector field $[X, Y]$ is defined by

$$[X, Y](f) = X(Y(f)) - Y(X(f)) \quad \text{for all } f \in C^\infty(M).$$

The operator $[\cdot, \cdot] : \mathcal{X}^\infty(M) \times \mathcal{X}^\infty(M) \to \mathcal{X}^\infty(M)$, called the **Lie bracket**, has the following properties:

(i) $[\cdot, \cdot]$ is bilinear.

(ii) $[X, Y] = -[Y, X]$ for all $X, Y \in \mathcal{X}^\infty(M)$, i.e., $[\cdot, \cdot]$ is antisymmetric.

(iii) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for all $X, Y, Z \in \mathcal{X}^\infty(M)$.

The third property is also called the **Jacobi identity**. The Lie bracket $[X, Y]$ is also defined for C^k-vector fields $X, Y \in \mathcal{X}^k(M)$ with $k \geq 1$. In this case, $[X, Y] \in \mathcal{X}^{k-1}(M)$ (see Lang [36, Proposition 1.3, p. 115]).

Tensor Fields

Let V be a d-dimensional vector space over $\mathbb{R}$. Then V^* denotes the dual space of V, $V^* = \mathrm{Hom}(V, \mathbb{R})$. The space of all multilinear mappings

$$t : \underbrace{V^* \times \cdots \times V^*}_{l \text{ factors}} \times \underbrace{V \times \cdots \times V}_{k \text{ factors}} \to \mathbb{R}$$

is denoted by $T^l_k(V)$ $(l, k \in \mathbb{N}_0)$. The elements of $T^l_k(V)$ are called **tensors** of type (l, k). Given $t_1 \in T^l_k(V)$ and $t_2 \in T^r_s(V)$, the **tensor product** of t_1 and t_2 is a tensor of type $(l + r, k + s)$, given by

$$\begin{aligned}(t_1 \otimes t_2)(\xi^1, \ldots,\ & \xi^{l+r}, \eta_1, \ldots, \eta_{k+s}) \\ &= \quad t_1(\xi^1, \ldots, \xi^l, \eta_1, \ldots, \eta_k) \cdot t_2(\xi^{l+1}, \ldots, \xi^{l+r}, \eta_{k+1}, \ldots, \eta_{k+s})\end{aligned}$$

for all $\xi^i \in V^*$ and $\eta_j \in V$, $i = 1, \ldots, l + r$, $j = 1, \ldots, k + s$.

If $\{e_1, \ldots, e_d\}$ is a basis of V and $\{e^1, \ldots, e^d\}$ its dual basis, then e_i $(i = 1, \ldots, d)$ can be interpreted as a tensor of type $(1, 0)$ by setting $e_i(\xi) := \xi(e_i)$ and e^j $(j = 1, \ldots, d)$ is by definition a tensor of type $(0, 1)$. A basis of $T^l_k(V)$ is given by the tensors

$$e_{i_1} \otimes \cdots \otimes e_{i_l} \otimes e^{j_1} \otimes \cdots \otimes e^{j_k},$$

where $i_1, \ldots, i_l, j_1, \ldots, j_k \in \{1, \ldots, d\}$ and hence $\dim T^l_k(V) = d^{k+l}$ (see Abraham & Marsden & Ratiu [1, Proposition 5.1.2, p. 339]). A tensor $t \in T^k_0(V)$ is called **symmetric** if

$$t(\xi^1, \ldots, \xi^k) = t(\xi^{\sigma(1)}, \ldots, \xi^{\sigma(k)})$$

and **skew-symmetric** if

$$t(\xi^1, \dots, \xi^k) = \operatorname{sign}(\sigma) t(\xi^{\sigma(1)}, \dots, \xi^{\sigma(k)})$$

for all vectors $\xi^1, \dots, \xi^k \in V^*$ and permutations $\sigma \in \Sigma_k$. Analogously one defines symmetric and skew-symmetric tensors of type $(0,k)$. By $\bigwedge^k V$ we denote the vector space of all skew-symmetric tensors of type $(k,0)$ and by $\bigwedge^k V^*$ the vector space of all skew-symmetric tensors of type $(0,k)$. The dimension of both $\bigwedge^k V$ and $\bigwedge^k V^*$ is $\binom{d}{k}$.

The **wedge product** of two tensors $t \in T^0_k(V)$ and $s \in T^0_l(V)$ is an element of $\bigwedge^{k+l} V^*$, defined by

$$(t \wedge s)(e_1, \dots, e_{k+l}) := \frac{1}{k!l!} \sum_{\sigma \in \Sigma_{k+l}} \operatorname{sign}(\sigma)(t \otimes s)(e_{\sigma(1)}, \dots, e_{\sigma(k+l)}).$$

The wedge product is bilinear, associative and it satisfies $t \wedge s = (-1)^{kl} s \wedge t$.

Let M be a d-dimensional smooth manifold. A $\boldsymbol{C^r}$**-tensor field** ($r \in \mathbb{N}_0 \cup \{\infty\}$) of type (l,k) on M is given by a family $t = (t_p)_{p \in M}$ of tensors $t_p \in T^l_k(T_pM)$, $p \in M$, such that the functions

$$t^{i_1,\dots,i_l}_{j_1,\dots,j_k} : U \to \mathbb{R}, \quad i_1, \dots, i_l, j_1, \dots, j_k \in \{1, \dots, d\},$$

defined by

$$t^{i_1,\dots,i_l}_{j_1,\dots,j_k}(p) := t_p(d\phi_{i_1}(p), \dots, d\phi_{i_l}(p), \partial_{j_1}\phi(p), \dots, \partial_{j_k}\phi(p))$$

are of class C^r. A C^r-tensor field t of type $(0,k)$ (or $(k,0)$) is called (skew-)symmetric if t_p is (skew-)symmetric for all $p \in M$. A C^r-tensor field of type $(0,0)$ is an element of $C^r(M)$. A C^r-tensor field of type $(1,0)$ can be identified canonically with a C^r-vector field. The tensor product and the wedge product for tensor fields are defined pointwise.

If $f : M \to N$ is a C^r-map ($r \geq 1$) between smooth manifolds M and N, and t is a C^r-tensor field on N of type $(0,k)$, the **pullback** f^*t of t by f is a C^{r-1}-tensor field on M of type $(0,k)$, defined by

$$(f^*t)(x)(v_1, \dots, v_k) := t(f(x))(Df_x v_1, \dots, Df_x v_k) \tag{A.10}$$

for all $x \in M$ and $v_1, \dots, v_k \in T_xM$ (cf. Abraham & Marsden & Ratiu [1, Definition 5.2.16, p. 354]).

Let $X \in \mathcal{X}^r(M)$, $r \geq 1$. The **Lie derivative** $\mathcal{L}_X$ assigns to each C^r-tensor field t of type (l,k) a C^{r-1}-tensor field $\mathcal{L}_X(t)$ of type (l,k). For $f \in C^r(M)$ $\mathcal{L}_X(f)$ is defined by $X(f)$. For $Y \in \mathcal{X}^r(M)$ the Lie derivative is given by $\mathcal{L}_X(Y) = [X,Y]$. $\mathcal{L}_X$ can be uniquely extended to a differential operator on the full tensor algebra of M (see Abraham & Marsden & Ratiu [1, Section 5.3]).

Riemannian Metrics

Let M be a d-dimensional smooth manifold. A **Riemannian metric** on M of class C^k ($k \in \mathbb{N}_0 \cup \{\infty\}$) is a symmetric and positive definite C^k-tensor field g of type $(0, 2)$ (i.e., $g_p : T_pM \times T_pM \to \mathbb{R}$ is symmetric and positive definite for every $p \in M$). The pair (M, g) is called a **Riemannian manifold** (of class C^k). If (ϕ, U) is a chart of M, then for every $p \in U$ g_p can be described by a symmetric positive definite matrix $(g_{ij}(p))_{1 \le i,j \le d}$, given by

$$g_{ij}(p) = g_p(\partial_i \phi_p, \partial_j \phi_p), \quad i, j = 1, \ldots, d.$$

The Riemannian metric g induces a norm $\| \cdot \|_p$ on T_pM for every $p \in M$ by

$$\|v\|_p := \sqrt{g_p(v, v)}.$$

The length of a (piecewise) C^1-curve $c : [a, b] \to M$ is defined by

$$\mathcal{L}(c) := \int_a^b \|\dot{c}(t)\|_{c(t)} dt.$$

Moreover, the Riemannian metric g induces a metric on M by

$$d(p, q) := \inf \left\{ \mathcal{L}(c) \mid c : [a, b] \to M \text{ piecewise } C^1,\ c(a) = p,\ c(b) = q \right\}.$$

The metric d is called the **Riemannian distance** on (M, g). The topology induced by d coincides with the given one.

A **connection** on M is a mapping $\nabla : \mathcal{X}^\infty(M) \times \mathcal{X}^\infty(M) \to \mathcal{X}^\infty(M)$, $(X, Y) \mapsto \nabla_X Y$, which satisfies the following axioms for all $X, Y, X_1, X_2, Y_1, Y_2 \in \mathcal{X}^\infty(M)$ and $f \in C^\infty(M)$:

(i) $\nabla_{X_1 + X_2} Y = \nabla_{X_1} Y + \nabla_{X_2} Y$.

(ii) $\nabla_{fX} Y = f \nabla_X Y$.

(iii) $\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$.

(iv) $\nabla_X (fY) = f \nabla_X Y + X(f) Y$.

On every Riemannian manifold (M, g) of class C^∞ there exists a unique connection ∇ which additionally satisfies the following axioms:

(i) $[X, Y] = \nabla_X Y - \nabla_Y X$.

(ii) $Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$.[9]

This connection is called the **Levi-Civita connection**. If (ϕ, U) is a chart of M, the Levi-Civita connection ∇ defines d^3 C^∞-functions $\Gamma^k_{ij} : U \to \mathbb{R}$, $i, j, k = 1, \ldots, d$, by

$$\left(\nabla_{\partial_i \phi} \partial_j \phi\right)_p = \sum_{k=1}^d \Gamma^k_{ij}(p) \partial_k \phi_p. \tag{A.11}$$

[9] $Z(g(X, Y))$ means: The vector field Z applied to the C^∞-function $p \mapsto g_p(X_p, Y_p)$.

The functions Γ_{ij}^k are called the **Christoffel symbols (of the second kind)** of (M, g) with respect to (ϕ, U). They satisfy the following equation:

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{l=1}^{d} g^{kl} \left(\frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l} \right), \tag{A.12}$$

where $(g^{ij}(p))_{1 \leq i,j \leq d}$ is the inverse of the positive definite symmetric matrix $(g_{ij}(p))_{1 \leq i,j \leq d}$, and

$$\frac{\partial g_{ij}}{\partial x_k} :\equiv \partial_k \phi(g_{ij}).$$

If $X \in \mathcal{X}^k(M)$ with $k \geq 1$, then the covariant derivative of X at $p \in M$ is the endomorphism $\nabla X(p) : T_pM \to T_pM$, defined by $v \mapsto (\nabla_v X)(p)$. This definition makes sense, since for fixed $X \in \mathcal{X}^k(M)$ the vector $(\nabla_Y X)(p)$ depends only on $Y(p)$.

A vector field along a C^∞-curve $c : I \to M$ is a function $X : I \to M$ with $X_t := X(t) \in T_{c(t)}M$ for all $t \in I$. It is called smooth, if for every $t_0 \in I$ and every chart (ϕ, U) with $c(t_0) \in U$ the functions $\xi^1, \ldots, \xi^d$, which satisfy $X(t) = \sum_i \xi^i(t) \partial_i \phi(c(t))$, are of class C^∞. The set of all smooth vector fields along c is denoted by $\mathcal{X}_c$. There exists a unique mapping $\frac{D}{dt} : \mathcal{X}_c \to \mathcal{X}_c$, which satisfies the following axioms for all $X, X_1, X_2 \in \mathcal{X}_c$, $f \in C^\infty(I)$ and $Y \in \mathcal{X}^\infty(M)$:

(i) $\frac{D(X_1+X_1)}{dt} = \frac{DX_1}{dt} + \frac{DX_2}{dt}$.

(ii) $\frac{D(fX)}{dt} = f'X + f\frac{DX}{dt}$.

(iii) $\frac{D(Y \circ c)}{dt} = \nabla_{\dot{c}(t)} Y$.

$\frac{D}{dt}$ is called the **covariant derivative along *c***. A further property of $\frac{D}{dt}$ is

$$\frac{d}{dt} g_{c(t)}(X(t), Y(t)) = g_{c(t)} \left(\frac{DX}{dt}(t), Y(t) \right) + g_{c(t)} \left(X(t), \frac{DY}{dt} \right). \tag{A.13}$$

A C^∞-curve $c : I \to M$ is called a **geodesic** if $\frac{D\dot{c}}{dt}(t) = 0$ for all $t \in I$. Every geodesic is parametrized proportionally to its arclength, i.e., $\|\dot{c}(t)\|$ is constant. For every $p \in M$ and $v \in T_pM$ there exists a unique open interval I with $0 \in I$ and a maximal geodesic $c_v : I \to M$ with $c(0) = p$ and $\dot{c}(0) = v$. That is, every geodesic $\tilde{c} : J \to M$ with $\tilde{c}(0) = p$ and $\dot{\tilde{c}}(0) = v$ is a restriction of c_v. The subset of T_pM, where $c_v(1)$ is defined, contains an open neighborhood U_p of 0, such that mapping $\exp_p : U_p \to M$, $\exp_p(v) := c_v(1)$, is a C^∞-diffeomorphism onto its image. In particular, it holds that

$$D \exp_p(0) = \mathrm{id}_{T_pM} . \tag{A.14}$$

$\exp_p$ is called the **Riemannian exponential map** at $p \in M$.

Every piecewise C^1-curve $c : [a, b] \to M$ with $\mathcal{L}(c) \leq \mathcal{L}(\tilde{c})$ for all $\tilde{c} : [a, b] \to M$ with $\tilde{c}(a) = c(a)$ and $\tilde{c}(b) = c(b)$, is a geodesic. On the other hand, for every $p \in M$ there exists $\varepsilon > 0$ such that for all $\delta \in [0, \varepsilon)$ and for every $v \in T_pM$ with $\|v\| = 1$ the geodesic $c_v : [0, \delta] \to M$ is the shortest curve between its endpoints.

By the *Theorem of Hopf-Rinow* the following assertions are equivalent for a Riemannian manifold (M, g):

(a) All maximal geodesics are defined on $\mathbb{R}$.

(b) There exists a point $p_0 \in M$ such that all maximal geodesics starting at p_0 are defined on $\mathbb{R}$.

(c) Every bounded and closed subset of M is compact.

(d) M is a complete metric space with the distance induced by g.

Let $\alpha : M \to \mathbb{R}$ be a C^1-mapping. Then the **gradient** of α is the unique continuous vector field on M, which is locally given by

$$\operatorname{grad} \alpha(x) := \sum_{i,j} g^{ij}(x) \partial_j \phi_x(\alpha) \partial_i \phi_x. \tag{A.15}$$

See also Abraham & Marsden & Ratiu [1, p. 354].

Volume Forms

Let M be a d-dimensional smooth manifold. The real vector space of all skew-symmetric C^r-tensor fields $\omega \in \mathcal{T}_k^0(M)$ on M is denoted by $\Omega_r^k(M)$. The elements of $\Omega_r^k(M)$ are called ***k*-forms**. The d-forms $\omega \in \Omega_r^d(M)$ with $\omega(p) \neq 0$ for all $p \in M$ are called **volume forms** (cf. Abraham & Marsden & Ratiu [1, Definition 6.5.1, p. 449]). If there exists a C^1-volume form ω on M, then M is called **orientable** and (M, ω) is called a **volume manifold**. In this case, any other volume ω' form on M can be written as $\omega' = f \cdot \omega$ for some $f \in C^1(M)$ (cf. Abraham & Marsden & Ratiu [1, Proposition 6.5.2, p. 449]).

The **standard volume form** ω_0 on $\mathbb{R}^d$ is given by

$$\omega_0(x) := dx^1 \wedge \cdots \wedge dx^d \quad \text{for all } x \in \mathbb{R}^d,$$

where $dx^i \in T_1^0(\mathbb{R}^d)$ is the linear map $v \mapsto \langle v, e_i \rangle$.

Let M and N be smooth d-dimensional manifolds and ω a C^r-volume form on N, $r \geq 1$. Let $f : M \to N$ be a C^r-diffeomorphism. Then $f^*\omega$ is a C^{r-1}-volume form on M.

Let ω be a C^1-volume form on M and $f : M \to M$ a C^1-map. Then for every $x \in M$ we define $\det_\omega Df : M \to \mathbb{R}$ by

$$(f^*\omega)(p) \equiv \det\nolimits_\omega Df(p) \cdot \omega(p). \tag{A.16}$$

For a vector field $X \in \mathcal{X}^1(M)$ the **divergence** with respect to the volume form ω is defined by the equation

$$(\mathcal{L}_X \omega)(p) \equiv \operatorname{div}_\omega X(p) \cdot \omega(p). \tag{A.17}$$

If $\alpha \in C^1(M)$ and $\alpha(p) \neq 0$ for all $p \in M$, then (cf. Abraham & Marsden & Ratiu [1, Proposition 6.5.17, pp. 455–456])

$$\operatorname{div}_{\alpha\omega} X = \operatorname{div}_\omega X + \frac{X(\alpha)}{\alpha}. \tag{A.18}$$

A C^r-volume form ω on a d-dimensional smooth manifold M induces a Borel measure μ_ω (and hence an integral) on M as follows: Let $A \subset M$ be a Borel set, which is contained in the domain V of a chart (ϕ, V) of M. Then the pullback $\tilde{\omega} := (\phi^{-1})^* \omega$ is a volume form on $\phi(V)$ and hence there exists a C^r-function $\alpha : \phi(V) \to \mathbb{R}$ with $\tilde{\omega}(x) = \alpha(x) \cdot \omega_0(x)$. Then the measure of A is defined by

$$\mu_\omega(A) := \int_{\phi(A)} |\alpha(x)| d\lambda^d(x). \tag{A.19}$$

This definition is independent of the chosen chart and μ_ω can be extended uniquely to a measure on the full Borel σ-algebra of M (see also Lang [36, Theorem 4.3, p. 301]).

If $f : M \to \mathbb{R}$ is an integrable function on M with respect to the integral induced by μ_ω, and $g : M \to M$ a C^1-diffeomorphism, then the transformation rule holds:

$$\int_{g(A)} f(x) d\mu_\omega(x) = \int_A f(g(y)) \cdot |\det\nolimits_\omega Dg(y)| d\mu_\omega(y). \tag{A.20}$$

If (M, g) is an oriented Riemannian manifold, then there exists a canonical volume form ω_g on M induced by the Riemannian metric g, called the **Riemannian volume**. It is locally given by

$$\omega_g(p) = \sqrt{\det g(p)}\, dx^1 \wedge \cdots \wedge dx^d \tag{A.21}$$

with respect to any chart of an oriented atlas. The divergence of a vector field $X \in \mathcal{X}^1(M)$ with respect to ω_g is given by (cf. Taylor [52, Proposition 3.1, p. 131])

$$\operatorname{div}_{\omega_g} X(p) = \operatorname{tr} \nabla X(p). \tag{A.22}$$

A.2 Fractal Dimension and Topological Entropy

Fractal Dimension

Recall that a subset $Z \subset X$ of a metric space (X, d) is called totally bounded if for every $\varepsilon > 0$ it can be covered with finitely many ε-balls, or equivalently, if it can be written as the union of sets with diameter less than ε. If (X, d) is complete, a set $Z \subset X$ is totally bounded if and only if $\operatorname{cl} Z$ is compact. For a totally bounded set Z let $N(\varepsilon, Z)$ denote the minimal number of ε-balls which are necessary to cover Z.

A.2.1 Definition (Fractal Dimension):
Let (X, d) be a metric space and $Z \subset X$ totally bounded. Then the **fractal dimension** *or* **upper box dimension** *of Z (with respect to the metric d) is defined as*

$$\dim_F(Z) := \limsup_{\varepsilon \searrow 0} \frac{\ln N(\varepsilon, Z)}{\ln 1/\varepsilon}.$$

The next lemma shows that the fractal dimension of Z does not depend on the space it is embedded in.

A.2.2 Lemma:
Let (X,d) be a metric space and $Z \subset X$ a totally bounded set. Let $\dim_F(Z;X)$ denote the fractal dimension of Z as a subspace of (X,d) and $\dim_F(Z;Z)$ the fractal dimension of Z as a subspace of (Z,d). Then $\dim_F(Z;X) = \dim_F(Z;Z)$.

Proof:
By $N(\varepsilon,Z;X)$ ($N(\varepsilon,Z;Z)$) we denote the minimal cardinality of a covering of Z with ε-balls in X (in Z). For given $\varepsilon > 0$ let $\mathcal{B} = \{B_\varepsilon(x_1),\ldots,B_\varepsilon(x_n)\}$, $x_i \in X$, be a minimal covering of Z with ε-balls in X, i.e., in particular $n = N(\varepsilon,Z;X)$. Then for every $i \in \{1,\ldots,n\}$ there exists some $z_i \in B_\varepsilon(x_i) \cap Z$, since otherwise $\mathcal{B}$ would not be minimal. Let $\widetilde{\mathcal{B}} := \{B_{2\varepsilon}(z_1),\ldots,B_{2\varepsilon}(z_n)\}$. Now take an arbitrary point $z \in Z$. Then there exists $i \in \{1,\ldots,n\}$ with $d(z,x_i) < \varepsilon$. It follows that

$$d(z,z_i) \le d(z,x_i) + d(x_i,z_i) < \varepsilon + \varepsilon = 2\varepsilon.$$

Hence, $\widetilde{\mathcal{B}}$ is a covering of Z consisting of n balls in Z of radius 2ε. This implies

$$N(2\varepsilon,Z;X) \le N(2\varepsilon,Z;Z) \le N(\varepsilon,Z;X).$$

Hence, for all $\varepsilon \in (0,1)$ it holds that

$$\frac{\ln N(2\varepsilon,Z;X)}{\ln(1/\varepsilon)} \le \frac{\ln N(2\varepsilon,Z;Z)}{\ln(1/\varepsilon)} \le \frac{\ln N(\varepsilon,Z;X)}{\ln(1/\varepsilon)}.$$

Using that $\ln(1/\varepsilon) = \ln(2) + \ln(1/(2\varepsilon))$ we obtain

$$\limsup_{\varepsilon\searrow 0} \frac{\ln N(2\varepsilon,Z;X)}{\ln(2)+\ln(1/(2\varepsilon))} \le \limsup_{\varepsilon\searrow 0} \frac{\ln N(2\varepsilon,Z;Z)}{\ln(2)+\ln(1/(2\varepsilon))} \le \dim_F(Z;X).$$

Since

$$\frac{\ln N(2\varepsilon,Z;X)}{\ln(2)+\ln(1/(2\varepsilon))} = \underbrace{\frac{\ln(1/(2\varepsilon))}{\ln(2)+\ln(1/(2\varepsilon))}}_{\to 1 \text{ for } \varepsilon\to 0} \cdot \frac{\ln N(2\varepsilon,Z;X)}{\ln(1/(2\varepsilon))},$$

we obtain $\dim_F(Z;X) \le \dim_F(Z;Z) \le \dim_F(Z;X)$. □

The following proposition summarizes elementary properties of the fractal dimension (see Boichenko & Leonov & Reitmann [8, Proposition 2.2.2, p. 200]).

A.2.3 Proposition:
Let (X,d) be a metric space.

(i) $\dim_F(Z_1) \le \dim_F(Z_2)$, *if $Z_1 \subset Z_2 \subset X$ are totally bounded sets.*

(ii) $\dim_F(\bigcup_{j\ge 1} Z_j) \ge \sup_{j\ge 1} \dim_F(Z_j)$, *where $Z_j \subset X$, $j = 1,2,\ldots$, are totally bounded sets.*

(iii) $\dim_F(\bigcup_{j=1}^k Z_j) = \max_{j=1,\ldots,k} \dim_F(Z_j)$, *if $Z_j \subset X$ are totally bounded sets, $j = 1,2,\ldots,k$.*

(iv) *If* (X', d') *is a second metric space and* $\phi : X \to X'$ *is a bi-Lipschitz map, then* $\dim_F(Z) = \dim_F(\phi(Z))$ *for any totally bounded set* $Z \subset X$.

(v) *If* $Z \subset X$ *is a totally bounded set, then* $\dim_F(\operatorname{cl} Z) = \dim_F(Z)$.

(vi) *If* (M, g) *is a* d*-dimensional compact Riemannian manifold, then* $\dim_F(M) = d$.

Topological Entropy

Let (X, d) be a metric space and $f : X \to X$ a uniformly continuous map. The iterates of f are defined inductively by $f^0 := \mathrm{id}_X$ and $f^{n+1} = f \circ f^n$ for all $n \in \mathbb{N}_0$. It can easily be verified that for every $n \in \mathbb{N}$ the following function defines a metric on X which is topologically equivalent to d:

$$d_{n,f}(x, y) := \max_{0 \le i \le n-1} d(f^i(x), f^i(y)).$$

A set $E \subset X$ is called (n, ε)*-separated* if for all $x, y \in E$ with $x \neq y$ it holds that $d_{n,f}(x, y) \geq \varepsilon$. A set $F \subset X$ (n, ε)*-spans* another set $K \subset X$ if for every $x \in K$ there exists $y \in F$ with $d_{n,f}(x, y) < \varepsilon$.[10]

For every compact set $K \subset X$ we denote by $r_{\mathrm{sep}}(n, \varepsilon, K, f)$ the cardinality of a maximal (n, ε)-separated subset of K. Moreover, we denote by $r_{\mathrm{span}}(n, \varepsilon, K, f)$ the cardinality of a minimal subset of X, which (n, ε)-spans K. We define

$$\begin{aligned} h_{\mathrm{span}}(\varepsilon, K, f) &:= \limsup_{n\to\infty} \frac{1}{n} \ln r_{\mathrm{span}}(n, \varepsilon, K, f), \\ h_{\mathrm{sep}}(\varepsilon, K, f) &:= \limsup_{n\to\infty} \frac{1}{n} \ln r_{\mathrm{sep}}(n, \varepsilon, K, f). \end{aligned}$$

With these definitions the following statements hold true:

(i) $r_{\mathrm{span}}(n, \varepsilon, K, f) \le r_{\mathrm{sep}}(n, \varepsilon, K, f) \le r_{\mathrm{span}}(n, \frac{\varepsilon}{2}, K, f) < \infty$.

(ii) If $\varepsilon_1 < \varepsilon_2$, then $h_{\mathrm{span}}(\varepsilon_1, K, f) \geq h_{\mathrm{span}}(\varepsilon_2, K, f)$ and $h_{\mathrm{sep}}(\varepsilon_1, K, f) \geq h_{\mathrm{sep}}(\varepsilon_2, K, f)$.

Hence, the following definitions make sense:

A.2.4 Definition:
The **topological entropy** *of* f *is defined by*

$$\begin{aligned} h_{\mathrm{top}}(K, f) &:= \lim_{\varepsilon \searrow 0} h_{\mathrm{span}}(\varepsilon, K, f) = \lim_{\varepsilon \searrow 0} h_{\mathrm{sep}}(\varepsilon, K, f), \\ h_{\mathrm{top}}(f) &:= \sup_{K \subset X} h_{\mathrm{top}}(K, f), \end{aligned}$$

where the supremum is taken over all nonvoid compact subsets of X.

[10] In Bowen [10], $d_{n,f}(x, y) > \varepsilon$ for separated sets and $d_{n,f}(x, y) \le \varepsilon$ for spanning sets is required. But for our purposes it is better to relax the strict inequality and vice versa. For the values of the topological entropy this makes no difference. For example, in Katok & Hasselblatt [33] spanning and separated sets are defined in the same way as we do.

In general, $h_{\text{top}}(f)$ and even $h_{\text{top}}(K, f)$ depends on the metric. If d_1 and d_2 are two different metrics on X (inducing the same topology) such that the identity $\text{id} : (X, d_1) \to (X, d_2)$ is uniformly continuous, then the corresponding topological entropies coincide. In particular, this is the case if X is compact.

Now consider a continuous semiflow $\Phi : \mathbb{R}_0^+ \times X \to X$ on X.[11] We denote the time-t-map $\Phi(t, \cdot) : X \to X$ by Φ_t, and we assume that Φ is uniformly continuous in the sense of the following definition, which is taken from Section 5 of Bowen [10].

A.2.5 Definition:
The semiflow $\Phi : \mathbb{R}_0^+ \times X \to X$ is called **uniformly continuous** *if for all $t_0 > 0$ the following holds:*

$$\forall \varepsilon > 0 : \ \exists \delta > 0 : \ \forall t \in [0, t_0], \ x, y \in X : \ d(x, y) < \delta \Rightarrow d(\Phi_t(x), \Phi_t(y)) < \varepsilon.$$

Now we can define the topological entropy of Φ in a way analogous to how we did for maps: For every positive time $T > 0$ we introduce a metric on X by

$$d_{T,\Phi}(x, y) := \max_{t \in [0,T]} d(\Phi_t(x), \Phi_t(y)).$$

A set $E \subset X$ is called (T, ε)-separated if for all $x, y \in E$ with $x \neq y$ one has $d_{T,\Phi}(x, y) \geq \varepsilon$, and a set $F \subset X$ (T, ε)-spans another set $K \subset X$ if for all $x \in K$ there is $y \in F$ with $d_{T,\Phi}(x, y) < \varepsilon$. Then $r_{\text{sep}}(T, \varepsilon, K, \Phi)$ denotes the maximal cardinality of a (T, ε)-separated set contained in K, and $r_{\text{span}}(T, \varepsilon, K, \Phi)$ the minimal cardinality of a set which (T, ε)-spans K. The quantities $h_{\text{span}}(\varepsilon, K, \Phi)$ and $h_{\text{sep}}(\varepsilon, K, \Phi)$ are then defined by

$$\begin{aligned} h_{\text{span}}(\varepsilon, K, \Phi) &:= \limsup_{T \to \infty} \frac{1}{T} \ln r_{\text{span}}(T, \varepsilon, K, \Phi), \\ h_{\text{sep}}(\varepsilon, K, \Phi) &:= \limsup_{T \to \infty} \frac{1}{T} \ln r_{\text{sep}}(T, \varepsilon, K, \Phi), \end{aligned}$$

and the topological entropy of Φ is given by

$$\begin{aligned} h_{\text{top}}(K, \Phi) &:= \lim_{\varepsilon \searrow 0} h_{\text{span}}(\varepsilon, K, \Phi) = \lim_{\varepsilon \searrow 0} h_{\text{sep}}(\varepsilon, K, \Phi), \\ h_{\text{top}}(\Phi) &:= \sup_{K \subset X \text{ compact}} h_{\text{top}}(K, \Phi). \end{aligned}$$

The following proposition relates the topological entropy of a semiflow to the topological entropy of its time-one-map.[12]

A.2.6 Proposition:
The topological entropy of the semiflow Φ equals the topological entropy of its time-one-map: $h_{\text{top}}(\Phi) = h_{\text{top}}(\Phi_1)$.

[11] That is, $\Phi(0, x) = x$ and $\Phi(t + s, x) = \Phi(t, \Phi(s, x))$ for all $x \in X$ and $t, s \in \mathbb{R}_0^+$.

[12] Note that usually the topological entropy of a flow is just defined as the topological entropy of its time-one-map.

Proof:
Fix a compact set $K \subset X$ and real numbers $T, \varepsilon > 0$. Let $F \subset X$ be a set which (T, ε)-spans K with respect to the semiflow Φ and define $n \in \mathbb{N}$ to be the greatest natural number such that $n - 1 \leq T$. Then for every $x \in K$ there is some $y \in F$ with $\max_{t \in [0,T]} d(\Phi_t(x), \Phi_t(y)) < \varepsilon$. Since $\Phi_j = (\Phi_1)^j$ for all $j \in \mathbb{N}_0$, this implies

$$d_{n,\Phi_1}(x, y) = \max_{0 \leq j \leq n-1} d((\Phi_1)^j(x), (\Phi_1)^j(y)) \leq \max_{t \in [0,T]} d(\Phi_t(x), \Phi_t(y)) < \varepsilon.$$

Thus, F (n, ε)-spans the set K with respect to the map Φ_1, which implies $r(n, \varepsilon, K, \Phi_1) \leq r(T, \varepsilon, K, \Phi)$. It follows that

$$\begin{aligned} h_{\text{span}}(\varepsilon, K, \Phi_1) &= \limsup_{n \to \infty} \frac{1}{n} \ln r(n, \varepsilon, K, \Phi_1) \leq \limsup_{n \to \infty} \frac{1}{n} \ln r(n, \varepsilon, K, \Phi) \\ &\leq \limsup_{T \to \infty} \frac{1}{T} \ln r(T, \varepsilon, K, \Phi) = h_{\text{span}}(\varepsilon, K, \Phi). \end{aligned}$$

Consequently, $h_{\text{span}}(\Phi_1) \leq h_{\text{span}}(\Phi)$. In order to show the converse inequality, let $T, \varepsilon > 0$ and choose $\delta = \delta(\varepsilon)$ according to Definition A.2.5 with $t_0 = 1$. Let $n \in \mathbb{N}$ be the smallest natural number such that $T \leq n - 1$ and let $F \subset X$ be a set which (n, δ)-spans K with respect to Φ_1. Then for every $x \in K$ there is some $y \in F$ such that $d_{n,\Phi_1}(x, y) < \delta$. For every $t \in [0, T]$ there are unique $j \in \{0, 1, \ldots, n - 1\}$ and $s \in [0, 1)$ such that $t = j + s$, which implies

$$\begin{aligned} d(\Phi_t(x), \Phi_t(y)) &= d(\Phi_s(\Phi_j(x)), \Phi_s(\Phi_j(x))) \\ &= d(\Phi_s((\Phi_1)^j(x)), \Phi_s((\Phi_1)^j(x))) < \varepsilon. \end{aligned}$$

Consequently, F is also (T, ε)-spanning the set K with respect to the semiflow Φ. Now for given $T > 0$ let $n = n(T)$ denote the smallest integer with $T \leq n-1$. Then it follows that

$$\begin{aligned} h_{\text{span}}(\varepsilon, K, \Phi) &= \limsup_{T \to \infty} \frac{1}{T} \ln r(T, \varepsilon, K, \Phi) \leq \limsup_{T \to \infty} \frac{1}{T} \ln r(n(T), \delta, K, \Phi_1) \\ &\leq \limsup_{n \to \infty} \frac{1}{n-2} \ln r(n, \delta, K, \Phi_1) \\ &= \limsup_{n \to \infty} \frac{1}{n} \ln r(n, \delta, K, \Phi_1) = h_{\text{span}}(\delta, K, \Phi_1). \end{aligned}$$

Thus, $h_{\text{top}}(K, \Phi) \leq h_{\text{top}}(K, \Phi_1)$ and $h_{\text{top}}(\Phi) \leq h_{\text{top}}(\Phi_1)$. □

The following result can be found in Bowen [10] as Theorem 15.

A.2.7 Proposition:
The topological entropy of a linear map $f : \mathbb{R}^d \to \mathbb{R}^d$ is given by

$$h_{\text{top}}(f) = \sum_{|\lambda_i| > 1} \ln |\lambda_i|,$$

where $\lambda_1, \ldots, \lambda_d$ are the eigenvalues of f, and $\mathbb{R}^d$ is equipped with a metric induced by a norm.

A.3 Technical Lemmas

Before we formulate and prove several technical lemmas, we introduce some notation:

Let (X, d) be a metric space and $K \subset X$ some subset. Then, for every $\varepsilon > 0$, the ε-neighborhood of K is defined by

$$N_\varepsilon(K) := \{x \in X \mid \exists y \in K : \ d(x, y) < \varepsilon\}.$$

That is, $N_\varepsilon(K)$ is the union of the open balls $B_\varepsilon(x)$, $x \in K$, and thus an open neighborhood of K. For a point $x \in X$ and a nonvoid set $A \subset X$ the distance from x to A is defined by

$$\operatorname{dist}(x, A) := \inf_{a \in A} d(x, a).$$

A.3.1 Lemma (Continuity of the dist-Function):
Let (X, d) be a metric space and $A \subset X$ nonvoid. Then the function

$$x \mapsto \operatorname{dist}(x, A), \quad X \to \mathbb{R}_0^+,$$

is continuous.

Proof:
For all $x, y \in X$ and $a \in A$ we have

$$\operatorname{dist}(x, A) \leq d(x, a) \leq d(x, y) + d(a, y).$$

Hence, $\operatorname{dist}(x, A) - d(x, y) \leq d(a, y)$, which implies

$$\operatorname{dist}(x, A) - d(x, y) \leq \inf \{d(y, a) \mid a \in A\} = \operatorname{dist}(y, A).$$

Hence, $\operatorname{dist}(x, A) - \operatorname{dist}(y, A) \leq d(x, y)$. By changing the roles of x and y we obtain

$$|\operatorname{dist}(x, A) - \operatorname{dist}(y, A)| \leq d(x, y),$$

which proves the assertion. □

A.3.2 Lemma (Existence of Precompact ε-Neighborhoods):
Let M be a smooth manifold and $d : M \times M \to \mathbb{R}_0^+$ a metric on M, which induces the given topology. Then for every nonvoid compact set $K \subset M$ there exists some $\varepsilon > 0$ such that $\operatorname{cl} N_\varepsilon(K)$ is compact.

Proof:
Since M is locally compact, for every $x \in K$ there exists a neighborhood $K_x \subset M$ of x such that $\operatorname{cl} K_x$ is compact. Since K is compact there are $x_1, \ldots, x_n \in K$ $(n \in \mathbb{N})$ with $K \subset \bigcup_{i=1}^n K_{x_i}$. Let $W := \bigcup_{i=1}^n \operatorname{cl} K_{x_i}$. Then, as the finite union of compact sets, W is a compact neighborhood of K. Assume to the contrary that for all $\varepsilon > 0$ there is some $x \in M$ with $\operatorname{dist}(x, K) < \varepsilon$ and $x \notin W$. Then there are sequences $(y_n)_{n \in \mathbb{N}}$ and $(z_n)_{n \in \mathbb{N}}$ with $y_n \in M \backslash W$, $z_n \in K$ and $d(y_n, z_n) < \frac{1}{n}$

for all $n \in \mathbb{N}$. By compactness of K we may assume that $z_n \to z$ for $n \to \infty$ with $z \in K$. Consequently, also $y_n \to z$ for $n \to \infty$. Let $i \in \{1, \dots, n\}$ such that $z \in K_{x_i}$. Then, for sufficiently large n we obtain $y_n \in K_{x_i} \subset W$ in contradiction to $y_n \in M \backslash W$. Hence, there exists some $\varepsilon > 0$ with $N_\varepsilon(K) \subset W$, which implies that $\operatorname{cl} N_\varepsilon(K) \subset W$ is compact. □

The next lemma is an immediate consequence of Abraham & Marsden & Ratiu [1, Theorem 5.5.7 and Proposition 5.5.8, pp. 379–380].

A.3.3 Lemma (Existence of Cut-off Functions):
Let M be a smooth manifold and $U_1, U_2 \subset M$ open sets with $\operatorname{cl} U_1 \subset U_2$. Then there exists a C^∞-function $\theta : M \to [0,1]$ such that $\theta(x) = 1$ for all $x \in U_1$ and $\theta(x) = 0$ for all $x \in M \backslash U_2$.[13]

The following result is known as the *Gronwall Lemma*. For a proof see Sontag [50, Lemma C.3.1, p. 346].

A.3.4 Lemma (Gronwall Lemma):
Let $I \subset \mathbb{R}$ be an interval, $c \geq 0$ and $\alpha, \mu : I \to \mathbb{R}_0^+$ two functions such that α is locally integrable and μ is continuous. Suppose that for some $\sigma \in I$

$$\mu(t) \leq c + \int_\sigma^t \alpha(s)\mu(s)ds \quad \text{for all } t \geq \sigma,\ t \in I.$$

Then, it holds that

$$\mu(t) \leq c e^{\int_\sigma^t \alpha(s)ds}.$$

A.3.5 Lemma:
Let $d, m \in \mathbb{N}$, I an interval and $D \subset \mathbb{R}^d$ open. Let $f : D \times \mathbb{R}^m \to \mathbb{R}^d$ and $\xi : I \to D$ be continuous and $u : I \to \mathbb{R}^m$ Lebesgue measurable. Then the mapping

$$h(t) := f(\xi(t), u(t)), \quad h : I \to \mathbb{R}^d,$$

is Lebesgue measurable. If there exists a compact set $U \subset \mathbb{R}^m$ such that $u(t) \in U$ for almost all $t \in I$, then h is locally integrable.

Proof:
We have $h = f \circ (\xi \times u)$, where $\xi \times u : I \to D \times \mathbb{R}^m$, $t \mapsto (\xi(t), u(t))$. Since f is continuous, it suffices to show that $\xi \times u$ is measurable. This can be done by showing that the preimage of every open set of the form $X \times U \subset D \times \mathbb{R}^m$ with open $X \subset D$ and $U \subset \mathbb{R}^m$ is measurable, since the family of these sets generate the Borel σ-algebra of $D \times \mathbb{R}^m$ (see Elstrodt [20, Folgerungen 4.2, p. 18]). We have

$$(\xi \times u)^{-1}(X \times U) = \{t \in I \mid (\xi(t), u(t)) \in X \times U\} = \xi^{-1}(X) \cap u^{-1}(U).$$

Since both ξ and u are measurable, $\xi^{-1}(X)$ and $u^{-1}(U)$ are measurable and thus also their intersection.

[13] A function with this property is called a *cut-off function*.

Now assume that $u(t) \in U$ for almost all $t \in I$ and let $J = [a,b] \subset I$ be a compact interval. Then $\|h(t)\|$ is bounded from above by $\max_{z\in\xi(J)\times U} \|f(z)\|$ for almost all $t \in J$, which implies that the integral $\int_a^b h(t)dt$ exists. $\square$

A.3.6 Lemma (Symmetrized Covariant Derivative):
Let (M,g) be a Riemannian manifold of class C^∞ and $f \in \mathcal{X}^1(M)$. Let (ϕ,U) be a chart of M and $f(x) = \sum_i f^i(x)\partial_i\phi_x$ on U. Then the local expression of the covariant derivative ∇f with respect to the chart (ϕ,U) is given by

$$\nabla_v f(x) = \sum_{i,k} \left[\partial_i\phi_x(f^k) + \sum_j \Gamma^k_{ij}(x) f^j(x)\right] v^i \partial_k\phi_x, \tag{A.23}$$

where $v = \sum_i v^i\partial_i\phi_x$. The local expression of the symmetrized covariant derivative $S\nabla f(x) = \frac{1}{2}[\nabla f(x) + \nabla f(x)^*]$ is given by the matrix $(s_{\mu\nu}(x))$ with the entries

$$2s_{\mu\nu} = \partial_\nu\phi(f^\mu) + \sum_{\theta,\kappa} \partial_\theta\phi(f^\kappa) g^{\mu\theta} g_{\kappa\nu} + \sum_{i,l} f^i g^{\mu l} \frac{\partial g_{\nu l}}{\partial x_i}. \tag{A.24}$$

Proof:
Let $v \in T_xM$ $(x \in U)$, $v = \sum_i v^i\partial_i\phi_x$. Then

$$\begin{aligned}
\nabla_v f(x) &= \sum_i v^i \nabla_{\partial_i\phi_x}\left(\sum_j f^j(x)\partial_j\phi_x\right) \\
&= \sum_{i,j} v^i \left[f^j(x)\nabla_{\partial_i\phi_x}\partial_j\phi_x + \partial_i\phi_x(f^j)\partial_j\phi_x\right] \\
&\overset{(A.11)}{=} \sum_{i,j} v^i\partial_i\phi_x(f^j)\partial_j\phi_x + \sum_{i,j} v^i f^j(x) \sum_k \Gamma^k_{ij}(x)\partial_k\phi_x \\
&= \sum_{i,k} v^i\partial_i\phi_x(f^k)\partial_k\phi_x + \sum_{i,j,k} \Gamma^k_{ij}(x) f^j(x) v^i \partial_k\phi_x.
\end{aligned}$$

This proves (A.23). Now let $C(x) \in \mathbb{R}^{d\times d}$ be the matrix representation of $\nabla f(x) : T_xM \to T_xM$ with respect to the chart (ϕ,U) and let $g(x) = (g_{ij}(x))$ be the symmetric positive definite matrix, defined by $g_{ij}(x) = g_x(\partial_i\phi_x, \partial_j\phi_x)$ for all $x \in U$. It is a well-known fact from linear algebra that the matrix representation of the adjoint operator $\nabla f(x)^*$ is then given by $g(x)^{-1}C(x)^T g(x)$. From (A.23) it follows that

$$[C(x)]_{\mu\nu} = \partial_\nu\phi_x(f^\mu) + \sum_j \Gamma^\mu_{\nu j}(x) f^j(x) \quad \text{for} \quad \mu,\nu = 1,\dots,d.$$

For brevity we suppress the argument x in the following. Then we obtain

$$\begin{aligned}
[g^{-1}C^T g]_{\mu\nu} &= \sum_{\theta,\kappa} g^{\mu\theta}\partial_\theta\phi(f^\kappa) g_{\kappa\nu} + \sum_\theta g^{\mu\theta} \sum_\kappa \left[\sum_i \Gamma^\kappa_{i\theta} f^i\right] g_{\kappa\nu} \\
&= \sum_{\theta,\kappa} \left[\partial_\theta\phi(f^\kappa) + \sum_i \Gamma^\kappa_{i\theta} f^i\right] g^{\mu\theta} g_{\kappa\nu}.
\end{aligned}$$

The matrix representation of $2S\nabla f(x)$ is then given by

$$\begin{aligned}
\left[C+g^{-1}C^Tg\right]_{\mu\nu} &= \partial_\nu\phi(f^\mu)+\sum_j \Gamma^\mu_{\nu j}f^j+\sum_{\theta,\kappa}\left[\partial_\theta\phi(f^\kappa)+\sum_i \Gamma^\kappa_{i\theta}f^i\right]g^{\mu\theta}g_{\kappa\nu}\\
&= \partial_\nu\phi(f^\mu)+\sum_{\theta,\kappa}\partial_\theta\phi(f^\kappa)g^{\mu\theta}g_{\kappa\nu}+\sum_i \Gamma^\mu_{\nu i}f^i+\sum_{\theta,\kappa,i}\Gamma^\kappa_{i\theta}f^ig^{\mu\theta}g_{\kappa\nu}\\
&= \partial_\nu\phi(f^\mu)+\sum_{\theta,\kappa}\partial_\theta\phi(f^\kappa)g^{\mu\theta}g_{\kappa\nu}\\
&\quad+\sum_i f^i\left[\Gamma^\mu_{i\nu}+\sum_{\theta,\kappa}\Gamma^\kappa_{i\theta}g^{\mu\theta}g_{\kappa\nu}\right]\\
&= \partial_\nu\phi(f^\mu)+\sum_{\theta,\kappa}\partial_\theta\phi(f^\kappa)g^{\mu\theta}g_{\kappa\nu}+\sum_{i,l}f^ig^{\mu l}\frac{\partial g_{\nu l}}{\partial x_i}.
\end{aligned}$$

The latter equality is proved by the following calculations:

$$\begin{aligned}
\Gamma^\mu_{i\nu}+\sum_{\kappa,\theta}\Gamma^\kappa_{i\theta}g^{\mu\theta}g_{\kappa\nu} &\overset{(A.12)}{=} \frac{1}{2}\sum_l g^{\mu l}\left(\frac{\partial g_{il}}{\partial x_\nu}+\frac{\partial g_{\nu l}}{\partial x_i}-\frac{\partial g_{i\nu}}{\partial x_l}\right)\\
&\quad+\frac{1}{2}\sum_{\kappa,\theta}\sum_l g^{\kappa l}\left(\frac{\partial g_{il}}{\partial x_\theta}+\frac{\partial g_{\theta l}}{\partial x_i}-\frac{\partial g_{i\theta}}{\partial x_l}\right)g^{\mu\theta}g_{\kappa\nu}\\
&= \frac{1}{2}\sum_l g^{\mu l}\left(\frac{\partial g_{il}}{\partial x_\nu}+\frac{\partial g_{\nu l}}{\partial x_i}-\frac{\partial g_{i\nu}}{\partial x_l}\right)\\
&\quad+\frac{1}{2}\sum_{\kappa,\theta,l}\left(\frac{\partial g_{il}}{\partial x_\theta}+\frac{\partial g_{\theta l}}{\partial x_i}-\frac{\partial g_{i\theta}}{\partial x_l}\right)g^{\mu\theta}g^{\kappa l}g_{\kappa\nu}.
\end{aligned}$$

Using that $\sum_\kappa g^{\kappa l}g_{\kappa\nu}=\sum_\kappa g^{l\kappa}g_{\kappa\nu}=\delta_{l\nu}$ we get

$$\begin{aligned}
\Gamma^\mu_{i\nu}+\sum_{\kappa,\theta}\Gamma^\kappa_{i\theta}g^{\mu\theta}g_{\kappa\nu} &= \frac{1}{2}\sum_l g^{\mu l}\left(\frac{\partial g_{il}}{\partial x_\nu}+\frac{\partial g_{\nu l}}{\partial x_i}-\frac{\partial g_{i\nu}}{\partial x_l}\right)\\
&\quad+\frac{1}{2}\sum_\theta g^{\mu\theta}\left(\frac{\partial g_{i\nu}}{\partial x_\theta}+\frac{\partial g_{\theta\nu}}{\partial x_i}-\frac{\partial g_{i\theta}}{\partial x_\nu}\right)
\end{aligned}$$

We rename the counter θ by l in the second sum, which leads to

$$\frac{1}{2}\sum_l g^{\mu l}\left(\frac{\partial g_{il}}{\partial x_\nu}+\frac{\partial g_{\nu l}}{\partial x_i}-\frac{\partial g_{i\nu}}{\partial x_l}+\frac{\partial g_{i\nu}}{\partial x_l}+\frac{\partial g_{l\nu}}{\partial x_i}-\frac{\partial g_{il}}{\partial x_\nu}\right)=\sum_l g^{\mu l}\frac{\partial g_{\nu l}}{\partial x_i}.$$

This finishes the proof of (A.24). □

For the proof of the following lemma see Davies [18, Lemma 1.21, p. 14].

A.3.7 Lemma (Subadditive Functions):
Let $f:(0,\infty)\to\mathbb{R}_0^+$ be a subadditive function, i.e.,

$$f(t+s)\le f(t)+f(s)\quad \textit{for all } t,s>0.$$

Suppose that f is bounded on an interval of the form $(0, t_0]$ with $t_0 > 0$. Then the limit $\lim_{t\to\infty} \frac{f(t)}{t}$ exists and equals $\inf_{t>0} \frac{f(t)}{t}$.

A.3.8 Lemma:
Let $A \in \mathbb{R}^{d\times d}$ and denote by $\alpha(A)$ the maximum of the real parts of all eigenvalues of A. Then it holds that

$$\forall \delta > 0 : \ \exists c > 0 : \ \forall t \geq 0 : \ \left\| e^{At} \right\| \leq c e^{(\alpha(A)+\delta)t},$$

where $\|\cdot\|$ denotes the operator norm induced by an arbitrary norm on $\mathbb{R}^d$.

Proof:
For given $\delta > 0$ define $B_\delta := A - (\alpha(A)+\delta)I$. Then all eigenvalues of B_δ have negative real parts, and hence, by Robinson [47, Theorem 5.1, p. 108], there exist constants $a > 0$ and $c \geq 1$ such that

$$\left\| e^{B_\delta t} \right\| \leq c e^{-ta} \quad \text{for all } t \geq 0.$$

Since $e^{B_\delta t} = e^{-(\alpha(A)+\delta)t} e^{At}$, this implies

$$\left\| e^{At} \right\| \leq c \underbrace{e^{-at}}_{\leq 1} e^{(\alpha(A)+\delta)t} \leq c e^{(\alpha(A)+\delta)t},$$

which proves the assertion. □

A.3.9 Lemma:
Let B be a compact metric space and $\pi : B \times \mathbb{R}^d \to B$, $\pi(b,x) = b$, a trivial vector bundle. Let $\pi_{\mathcal{W}} : \mathcal{W} \to B$, $\mathcal{W} \subset B \times \mathbb{R}^d$, $\mathcal{W} = \bigcup_{b\in B} \{b\} \times W(b)$, be a k-dimensional subbundle. For every $b \in B$ let $P_b \in \mathbb{R}^{d\times d}$ denote the orthogonal projection onto $W(b)$. Then the mapping

$$b \mapsto P_b, \quad B \to \mathbb{R}^{d\times d},$$

is continuous. If $\mathcal{V} \subset B \times \mathbb{R}^d$, $\mathcal{V} = \bigcup_{b\in B} \{b\} \times V(b)$, is a second subbundle such that $B \times \mathbb{R}^d = \mathcal{W} \oplus \mathcal{V}$, then also the mapping

$$b \mapsto Q_b, \quad B \to \mathbb{R}^{d\times d},$$

is continuous, where Q_b denotes the projection onto $W(b)$ with respect to the decomposition $\mathbb{R}^d = W(b) \oplus V(b)$.

Proof:
Let $b_0 \in B$. Then, by definition of vector bundles (see, e.g., Colonius & Kliemann [16, Definition B.1.11, p. 534]), there exists an open neighborhood $U \subset B$ of b_0 and a homeomorphism $\varphi : \pi_{\mathcal{W}}^{-1}(U) \to U \times \mathbb{R}^k$ of the form $\varphi(b,x) = (b, \hat{\varphi}(b,x))$. Hence, for every $b \in U$ and $y \in \mathbb{R}^k$ there exists a unique $x \in W(b)$ with $\hat{\varphi}(b,x) = y$. In particular, the map $\hat{\varphi}_b : W(b) \to \mathbb{R}^k$, $x \mapsto \hat{\varphi}(b,x)$, is a homeomorphism, and it holds that

$$W(b) = \hat{\varphi}_b^{-1}(\mathbb{R}^k) \quad \text{for every } b \in U.$$

Now let $\{e_1(b_0),\dots,e_k(b_0)\}$ be an orthonormal basis of $W(b_0)$ and define $e_1(b),\dots,e_k(b)\in W(b)$ by

$$e_j(b) := \hat{\varphi}_b^{-1}(\hat{\varphi}_{b_0}(e_j(b_0)))$$

for every $b\in B$. Since $(b,y)\mapsto \hat{\varphi}_b^{-1}(y)$ is continuous, and k linearly independent vectors stay linearly independent under small perturbations, there exists a neighborhood $V\subset U$ of b_0 such that $\{e_1(b),\dots,e_k(b)\}$ is a basis of $W(b)$ for all $b\in V$. For every $b\in V$ let $\{\hat{e}_1(b),\dots,\hat{e}_k(b)\}$ be the orthonormal basis of $W(b)$, which results from the Gram-Schmid process applied to $\{e_1(b),\dots,e_k(b)\}$. Then $\hat{e}_j(b)$ depends continuously on $b\in V$, and the orthogonal projection P_b, $b\in V$, can be written as

$$P_b x = \sum_{j=1}^{k} \langle x, \hat{e}_j(b)\rangle\, \hat{e}_j(b).$$

Hence,

$$\begin{aligned}\|P_b - P_{b_0}\| &= \max_{\|x\|=1} \|(P_b - P_{b_0})x\| = \max_{\|x\|=1} \|P_b x - P_{b_0} x\| \\ &= \max_{\|x\|=1} \underbrace{\left\| \sum_{j=1}^{k} \left[\langle x, \hat{e}_j(b)\rangle\, \hat{e}_j(b) - \langle x, \hat{e}_j(b_0)\rangle\, \hat{e}_j(b_0)\right] \right\|}_{=:f(b,x)}.\end{aligned}$$

Since $f(b,x)$ is uniformly continuous on the compact set $W\times S^{d-1}$, where $W\subset V$ is a compact neighborhood of b_0 and $S^{d-1}=\{x\in\mathbb{R}^d \mid \|x\|=1\}$, for every $\varepsilon>0$ we find $\delta>0$ such that $d(b,b_0)<\delta$ implies $|f(b,x)-f(b_0,x)|<\varepsilon$ for all $x\in S^{d-1}$. This implies continuity of $b\mapsto P_b$ at b_0.
If $\mathcal{V}$ is another subbundle of $B\times\mathbb{R}^d$ with $B\times\mathbb{R}^d=\mathcal{W}\oplus\mathcal{V}$, we can analogously find a neighborhood $W\subset B$ of b_0, where bases $\{e_1(b),\dots,e_k(b)\}$ of $W(b)$ and $\{e_{k+1}(b),\dots,e_d(b)\}$ of $V(b)$, depending continuously on b, are defined. Then for each $(b,x)\in W\times\mathbb{R}^d$ there are unique $\alpha_1(b,x),\dots,\alpha_d(b,x)\in\mathbb{R}$ such that

$$x = \underbrace{\sum_{i=1}^{k} \alpha_i(b,x) e_i(b)}_{=Q_b x} + \sum_{i=k+1}^{d} \alpha_i(b,x) e_i(b).$$

Let $a_{ij}(b) := \langle e_i(b), e_j(b)\rangle$ for all $b\in W$ and $i,j=1,\dots,d$. Then $A(b) := (a_{ij}(b))_{1\le i,j\le d}$ is a symmetric positive definite matrix and for $j=1,\dots,d$

$$x_j(b) := \langle x, e_j(b)\rangle = \sum_{i=1}^{d} a_{ij}(b)\alpha_i(b,x).$$

Hence, the vectors $\hat{x} := (x_1(b),\dots,x_d(b))$ and $\alpha(b,x) := (\alpha_1(b,x),\dots,\alpha_d(b,x))$ satisfy $\hat{x} = A(b)\alpha(b,x)$, which implies $\alpha(b,x) = A(b)^{-1}\hat{x}$. Therefore, in particular $\alpha_1(b,x),\dots,\alpha_k(b,x)$ depend continuously on (b,x) and thus also $Q_b x$. Continuity of Q_b then follows by uniform continuity on compact sets. $\square$

Notation

SET THEORY

$\emptyset$	the empty set
$\{x \in X \mid E(x)\}$	set of all $x \in X$ with property $E(x)$
$\{x \in X : E(x)\}$	see above
$x \in A$	x element of A
$x \notin A$	x not an element of A
$A \subset B$	A subset of B
$A \supset B$	A superset of B
$A \cup B$	union of A and B
$\bigcup_{i \in I} A_i$	union of the sets A_i, $i \in I$
$A \cap B$	intersection of A and B
$\bigcap_{i \in I} A_i$	intersection of the sets A_i, $i \in I$
$A \backslash B$	A minus B
A^c	complement of the set A
$A \times B$	Cartesian product of A and B
$\#A$	cardinality of the set A

SPECIAL SETS

$\mathbb{N}$	set of natural numbers, $\{1, 2, 3, \dots\}$
$\mathbb{Z}$	set of integers, $\{\dots, -2, -1, 0, 1, 2, \dots\}$
$\mathbb{R}$	set of real numbers
$\mathbb{R}_0^+$	set of nonnegative real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathbb{R}^{n \times m}$	set of real matrices with n rows and m columns
$[a, b]$	closed interval
(a, b)	open interval
$[a, b)$	right-open interval
$(a, b]$	left-open interval
S^d	the d-dimensional sphere
$\mathbb{P}^d$	the d-dimensional real projective space

FUNCTIONS

$f : X \to Y$	function from X to Y
f^{-1}	inverse of the invertible function f
$f(A)$	image of the set A under f
$f^{-1}(B)$	preimage of the set B under f
$f\|_A$	restriction of f to the set A
id_X	the identity on X
$f \circ g$	composition of f and g
f^n	n^{th} iterate of the function $f : X \to X$
$\mathbb{1}_A$	characteristic function of the set A
$f(x) \equiv g(x)$	$f(x) = g(x)$ for all x

TOPOLOGICAL AND METRIC SPACES

$\mathrm{cl}\, A$	closure of the set A
$\mathrm{int}\, A$	interior of the set A
$\mathrm{bd}\, A$	boundary of the set A
(X, d)	metric space
$B_\varepsilon(x)$	open ball with radius ε centered at x
$N_\varepsilon(Q)$	ε-neighborhood of the set Q
$N(\varepsilon, K)$	minimal number of ε-balls needed to cover the set K
$\mathrm{dist}(x, A)$	distance of the point x to the set A, $\inf_{a \in A} d(x, a)$
$\dim_F(K)$	fractal dimension of the (totally bounded) set K
$\mathrm{diam}(A)$	diameter of the set A, $\sup_{x,y \in A} d(x, y)$
$(x_n)_{n \in \mathbb{N}}$	the sequence $n \mapsto x_n$
$x_n \to x$	the sequence $(x_n)_{n \in \mathbb{N}}$ is converging to x
$\lim_{n \to \infty} x_n = x$	see above
$C^0(X, Y)$	set of continuous function from X to Y

LINEAR ALGEBRA

$\langle \cdot, \cdot \rangle$	standard scalar product on $\mathbb{R}^d$
$\\| \cdot \\|$	Euclidean norm on $\mathbb{R}^d$
$e_1, \ldots, e_d$	standard basis vectors of $\mathbb{R}^d$
$U^\perp$	orthogonal complement of the subspace U
$U \oplus V$	internal sum of the linear subspaces U and V
$\langle v_1, \ldots, v_n \rangle$	linear span of $v_1, \ldots, v_n$
$\det(A)$	determinant of the matrix A
$\sigma(A)$	spectrum of the matrix A
$\ker(A)$	kernel of the matrix A
$\mathrm{im}(A)$	image of the matrix A
$\mathrm{tr}(A)$	trace of the matrix A
I	the identity matrix

L^*	the adjoint of a linear map L on a Euclidean vector space
$\mathrm{Gl}(d,\mathbb{R})$	the general linear group of $d \times d$-matrices
$\mathrm{Sym}(d,\mathbb{R})$	the space of real symmetric $d \times d$-matrices
$\mathrm{Hom}(V,W)$	the space of homomorphisms from V to W
$O(d)$	the orthogonal group of $\mathbb{R}^d$
V^*	dual space of V
$T^l_k(V)$	set of tensors of type (l,k) over the vector space V

DIFFERENTIABLE MANIFOLDS

$\dim(M)$	dimension of the manifold M
$C^k(M)$	the set of all C^k-mappings $f : M \to \mathbb{R}$ $(k \in \mathbb{N}_0 \cup \{\infty\})$
$C^\infty(M,p)$	the set of all function germs at $p \in M$
T_pM	tangent space at $p \in M$
$\partial_i\phi$	i^{th} basis vector associated with the chart (ϕ,U)
TM	tangent bundle of M
T^*M	cotangent bundle of M
$Df(p)$	differential of the map f at p
Df_p	see above
$\mathcal{X}^k(M)$	set of C^k-vector fields on M $(k \in \mathbb{N}_0 \cup \{\infty\})$
$\Omega^k_r(M)$	set of k-forms of class C^r on M
f^*t	pullback of a tensor t via f
$\mathcal{L}_X$	Lie derivative with respect to $X \in \mathcal{X}^k(M)$
div_ω	divergence operator associated with the volume form ω
$\det_\omega$	Jacobian determinant with respect to the volume form ω
μ_ω	measure induced by the volume form ω

RIEMANNIAN MANIFOLDS

(M,g)	Riemannian manifold
g_{ij}	components of the metric tensor with respect to a chart
g^{ij}	components of the inverse of (g_{ij})
Γ^k_{ij}	Christoffel symbols
$\exp_p$	Riemannian exponential map at $p \in M$
$\nabla X(p)$	covariant derivative of $X \in \mathcal{X}^k(M)$ at $p \in M$
$\mathrm{grad}\,\alpha(p)$	gradient of a C^1-function $\alpha : M \to \mathbb{R}$ at $p \in M$

TOPOLOGICAL ENTROPY

$d_{n,f}(x,y)$	$\max_{0\le j\le n-1} d(f^j(x),f^j(y))$
$r_{\mathrm{span}}(n,\varepsilon,K,f)$	minimal cardinality of an (n,ε)-spanning set for K
$r_{\mathrm{sep}}(n,\varepsilon,K,f)$	maximal cardinality of an (n,ε)-separated subset of K
$h_{\mathrm{top}}(K,f)$	topological entropy of $f\|_K$

$h_{\mathrm{top}}(f)$	topological entropy of the map f

MISCELLANEOUS

$\Sigma_{\mathrm{Ly}}(\cdot)$	Lyapunov spectrum
$\Sigma_{\mathrm{Mo}}(\cdot)$	Morse spectrum
$L^p(X,\mathbb{R}^m)$	the space of L^p-functions from X to $\mathbb{R}^m$
Σ_k	the symmetric group on the set $\{1,\ldots,k\}$
λ^d	d-dimensional Lebesgue measure
$\lvert z\rvert$	absolute value of a real or complex number z
$\mathrm{Re}(z)$	real part of the complex number z
$\mathrm{Im}(z)$	imaginary part of the complex number z
$\lfloor x\rfloor$	the greatest integer less or equal to $x \in \mathbb{R}$
$\nabla f(x)$	gradient of $f:\mathbb{R}^d \supset D \to \mathbb{R}$ at x
$\sup A$	supremum of a set $A \subset \mathbb{R}$
$\inf A$	infimum of a set $A \subset \mathbb{R}$
$\limsup_{t\to t_0} f(t)$	limes superior of f for $t \to t_0$
$\liminf_{t\to t_0} f(t)$	limes inferior of f for $t \to t_0$
$\lim_{t\to t_0} f(t)$	limit of f for $t \to t_0$
$\mathrm{supp}\, f$	support of a function $f : X \to \mathbb{R}$
exp	exponential function
$\log_a$	logarithm to the base a
ln	natural logarithm
sin	sine function
cos	cosine function
δ_{ij}	the Kronecker delta
$\dot\lambda(t)$	derivative of a curve $\lambda : I \to M$
$f'(x)$	first derivative of a function $f:(a,b)\to\mathbb{R}$ at x
$f''(x)$	second derivative of a function $f:(a,b)\to\mathbb{R}$ at x
$X \cong Y$	X isomorphic to Y

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